constant throughout any given trajectory; accents represent time-derivatives. As the earth is not flat, these equations must be altered so as to relate to a single coördinate system, in place of an infinite number of instantaneous systems. Captain Hoar pointed out that the "tangent method" uses cartesian axes, horizontal and vertical at the origin, *i.e.*, the gun. The "curved method" measures $x$ from the gun along a circle concentric with the earth, and $y$ vertically upward from this circle. In each method, the equations are as above, plus terms in $1/r$ and higher powers, $r$ being the radius of the earth. In practice, these terms may be omitted, the slight resulting error being absorbed in the value chosen for $C$.

6. The parametric equations of the space curves into which plane curves are twisted when their plane is developed on any cylinder or on any right circular cone are derived in the paper by Professor Luck. The purpose of this is to afford to students beginning the study of space curves a means of obtaining, by elementary means, the equations of certain classes of space curves rough models of which are easily available. In the case of the cylinder it is shown that if the arc length is the parameter used for the plane curve the equations of the space curve may be written down with the same parameter. The inverse problem of finding the equation of the plane curve into which a curve on the cylinder will pass when the cylinder is unrolled on the plane is also considered.

8. In this paper Dr. Murnaghan discusses the foundation of Newtonian dynamics, especially of the definitions of absolute systems of reference and vectors, Einstein's law of inertia and his hypothesis as to the nature of space which is sometimes called his "law of gravitation," and an application to the motion of a particle in the neighborhood of a single body.

10. If $f(x, y) = 0$ is the plane path of a particle, then in the equation:

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y'y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = 0$$

the second fraction is equal to $\alpha \cos\theta$, where $\alpha$ is the magnitude of the total acceleration and $\theta$ is the angle between the normal and the direction of acceleration. If then *e.g.*, $f(x, y)$ is taken as $b^2x^2 + a^2y^2 - a^2b^2$, and the acceleration is directed toward a focus, Dr. Cresse shows that the classical expressions for $\alpha$ and $v$ in planetary motion are read off. If for a parabola of higher degree,

$$f(x, y) \equiv y - (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n) \equiv y - \phi(x),$$

and the acceleration is composed of $g$ taken vertically and of $\alpha_T$, the additional acceleration along the tangent, then

$$v = \frac{\sqrt{g}}{[-\varphi'(x)]^{1/2}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \qquad \alpha_T = v\frac{\sqrt{g}}{2}\frac{\varphi'''(x)}{[-\varphi''(x)]^{3/2}}.$$

11. Professor Echols's "Note" was published in the last issue of this MONTHLY.

O. S. ADAMS, *Secretary-Treasurer.*

## THE APRIL MEETING OF THE IOWA SECTION.

The sixth regular meeting of the Iowa Section of the Mathematical Association of America was held at the University of Iowa, Iowa City, on Saturday morning, April 24, 1920. Twenty-five persons were in attendance, including the following twenty members:

R. P. Baker, J. W. Campbell, E. W. Chittenden, L. M. Coffin, Julia Calpitts, I. S. Condit, Marian Daniells, Fay Farnum, M. E. Graber, R. B. McClenon, F. M. McGaw, J. V. McKelvey, M. A. Nordgaard, J. F. Reilly, H. L. Rietz, Maria Roberts, B. F. Simonson, F. M. Weida, C. W. Wester, W. H. Wilson.

The following officers were elected for 1920–21: Chairman: Professor H. L. RIETZ, Univ. of Iowa; Vice-chairman: Professor R. B. MCCLENON, Grinnell College; Secretary-Treasurer: B. F. SIMONSON, Upper Iowa University.

The following papers were read:

(1) "Notes from the history of indeterminate analysis" by Professor R. B. MCCLENON, Grinnell College;
(2) "The taxonomy of algebraic surfaces" by Professor R. P. BAKER, State University;
(3) "What is number?" by Professor C. W. WESTER, State Teachers College;
(4) "A special form of standardization trajectory" by Professor M. E. GRABER, Morningside College;
(5) "Note on a generalization of a theorem of Baire" by Professor E. W. CHITTENDEN, State University;
(6) "The integration of the indefinite integral in the first course" by Dr. W. H. WILSON, State University;
(7) "Limits in secondary mathematics" by Professor J. V. MCKELVEY, State College;
(8) "A problem in summation of series" by Professor J. F. REILLY, State University;
(9) "Geometric construction for the regular 17-gon" by LINN SMITH,[1] student at Grinnell College. (Presented in his absence by Professor McClenon.)

Professor Rietz presented the critical situation in the Iowa colleges and universities of students entering without mathematics, a condition expected soon to arise because of a recent law admitting to the three state institutions, without examination, graduates of any approved secondary institution who present sixteen units. A general discussion followed.

L. M. COFFIN, *Secretary-Treasurer.*

[1] Mr. Smith's paper was printed in the last issue of this MONTHLY.

# THE AVERAGE READING VOCABULARY; AN APPLICATION OF BAYES'S THEOREM.

By WARREN WEAVER, University of Wisconsin.

The rule for the computation of *a posteriori* probabilities was first developed by an English clergyman, T. Bayes, and was published after his death in the *Philosophical Transactions* for 1763. The careless application of this rule has led to many paradoxical results,[1] in consequence of which some mathematicians would abandon the rule entirely. Among this number may be mentioned Mr. J. Bing, a Danish actuary, the late Dr. T. Thiele, and especially Professor Chrystal, whose advice is to "bury the laws of inverse probability decently out of sight." The problem herein stated and solved may be of interest since it clearly emphasizes the point the neglect of which has led to incorrect results, since it shows what great allowances may sometimes be made in the *a priori* probabilities of existence and still allow us to change our view in regard to a statistical result, and since it is an answer to a specific case of that interesting question, to what degree does new experimental evidence justify us in modifying previously held opinions.

1. **Statement of the theorem.** Some one of the mutually exclusive causes $A_1, A_2, \cdots A_n$ is to produce an event. When the result is not known (*i.e.*, before the event occurs) the existence probability for each cause is $\pi_1, \pi_2, \cdots \pi_n$ (*i.e.*, the *a priori* probability of the existence of each cause). The event in question occurs. The cause $A_1$, when it is known to act, gives the productive probability $p_1$, etc. Then the *a posteriori* probability that the cause $A_r$ produced the event is

$$P_r = \frac{p_r\pi_r}{p_1\pi_1 + p_2\pi_2 + \cdots + p_r\pi_r + \cdots + p_n\pi_n}. \tag{1}$$

If the event in question is able to occur in two alternative ways one of which we call "successful," and the other of which "unsuccessful"; and if, further, the productive probability that the cause $A_r$ produce the event successfully be $\omega_r$, then if the event occurs successfully $m$ times in $k$ trials the *a posteriori* probability that the cause $A_r$ has been the one to act is

$$P_r = \frac{\pi_r\omega_r{}^m(1-\omega_r)^{k-m}}{\Sigma\pi_i\omega_i{}^m(1-\omega_i)^{k-m}} \quad (i = 1, 2, \cdots n). \tag{2}$$

The theorem may assume a third form in problems of such a nature that the different causes $A_r$ may be considered different stages of a continuously changing complex. In this case the quantities involved in the formula become definite integrals.[2]

[1] For example, Bing's Paradox: "If among a large group of $S$ equally old persons we have observed no deaths during a full calendar year, then another person of the same age outside the group is certain to die inside the calendar year" quoted from *The Mathematical Theory of Probabilities*, by A. Fisher. Volume 1, New York, 1915, p. 75.

[2] A. Fisher, *l.c.*, p. 67.

To those not entirely familiar with the theorem an example may make the statement of it more clear. Suppose that we have an urn filled with black and white balls in unknown proportion, and that our *a priori* estimate of the existence probability that there are $x$ white and $(b - x)$ black is $\pi_x$ ($b$ being the total number of balls in the urn). Suppose that we draw $k$ balls from the urn, returning each, and find that of these $m$ are white and $k - m$ black. What is then the most probable mixture in the urn? It will not be the one originally most probable, that is, the one for which $\pi_x$ is a maximum; nor will it be the one suggested by the drawing,[2] but will obviously be some mixture intermediate between these two. It will be, in fact, a mixture of $x^*$ white and $(b - x^*)$ black, where $x^*$ is that value of $x$ for which

$$P_x = \frac{\pi_x \left(\frac{x}{b}\right)^m \left(\frac{b-x}{b}\right)^{k-m}}{\Sigma \pi_{x'} \left(\frac{x'}{b}\right)^m \left(\frac{b-x'}{b}\right)^{k-m}} \quad (x' = 1, 2, \cdots n) \tag{3}$$

is a maximum. In case $\pi_x$ is given, by experiment, judgment, or calculation, for a certain finite number of values of $x$, we might determine this value $x^*$ by plotting to any scale whatsoever, and noting the value of $x$ corresponding to the highest point on the curve. If we should later wish the vertical scale of this curve we could most easily determine it from the fact that the area under it must equal unity.

2. **Statement of the problem.** A test has been devised by the department of educational psychology at the University of Wisconsin to determine a person's reading vocabulary. The process consists of taking at random 200 words from the dictionary, and having a person decide with how many of the 200 words he is familiar—say 117. Then the value of this person's reading vocabulary is taken as

$$(117/200)\ 104{,}000$$

or about 61,000. (104,000 being the approximate number of words in the dictionary.)

The scheme has been found to give, as the result of about five hundred tests by university students, the value given above. This value is far in excess of previous estimates, the general opinion before this test being, according to Professor Starch, that the correct figure was in the neighborhood of 25,000. We wish to investigate whether this process gives us a sound basis for raising our previous estimate of 25,000 to 61,000, and if not, what our answer should be.

Since, as will appear later, the *a priori* probabilities of different estimates have an important effect upon the solution of the problem it is necessary to inquire as carefully as the nature of the existing information will permit into the basis and reliability of this estimate of 25,000 words. Unfortunately the information is vague, but it appears to be an average opinion of those who had been interested in the matter, rather than a definite statistical result. There may have been, to be sure, some numerical method, however unsatisfactory, by which the

[2] Unless, of course, these two mixtures happen to be the same.

estimates were arrived at; or it may be that an investigator with a great deal of experience along this line would venture an estimate based upon intuition alone. It is evident as a mere matter of common sense that if this estimate of 25,000 words were formed upon the basis of one application of this test itself, and the next application of the test furnished an estimate of 61,000, one would not be justified in completely discarding the old estimate for the new. It is obviously a matter of the comparative reliability of the new and old judgments, which comparison is made accurately by the theorem stated.

3. **Solution of the problem.** The actual method used in taking the sample was to take the first word on the $k$th page of a Webster's *Unabridged Dictionary*, $k$ having such a value that the method would result in a sample of 200. There being so many words to pick from, it is evident that it is immaterial whether or not we consider that we return each word after its drawing: a consideration which in other cases might be important. It seems likely, on an intuitive basis, that if the same person performed the test several times with different samples, or if it were performed with several persons and the same sample, results would be obtained that would vary widely, especially since 200 seems a small sample from a group of 104,000. It should be emphasized therefore that the datum which we use is the average of over five hundred results from different persons. And some knowledge of the variability of these results is important in making an estimate of the stability of the average. The following frequency table gives us an estimate of the variability in the results obtained from a typical group of fifty students, using the same sample of words. It is on the basis of 100 words rather than 200 since, as a matter of proceedure in making the test, the whole list was split up into two lists of 100 each, and the score kept for each separately.

| No. of Words Known. | No. of Students. | No. of Words Known. | No. of Students. |
|---|---|---|---|
| 46 | 0 | 61 | 1 |
| 47 | 0 | 62 | 3 |
| 48 | 1 | 63 | 2 |
| 49 | 1 | 64 | 3 |
| 50 | 2 | 65 | 0 |
| 51 | 0 | 66 | 1 |
| 52 | 3 | 67 | 3 |
| 53 | 1 | 68 | 4 |
| 54 | 2 | 69 | 0 |
| 55 | 3 | 70 | 5 |
| 56 | 1 | 71 | 0 |
| 57 | 3 | 72 | 1 |
| 58 | 4 | 73 | 0 |
| 59 | 1 | 74 | 0 |
| 60 | 5 | ⋮ | — |
| | | | 50 |

The result shown is typical of all obtained, a mean variation of approximately five being found among all the students tested. We conclude therefore that 117 is an average of considerable stability, coming as it does from a group of over 500 tests which show a relatively small variability. Unfortunately the data are not available as to how many sets were used: the errors of sampling in the dictionary would be reduced in the approximate ratio of 1 to $\sqrt{n}$ where $n$ is the number of sets used. However the accuracy of any one list as expressed by the mean variation between two lists of 100 words each is between two and three words. We are led then to this conclusion: that if we imagine the hypothetical case of "the average" university student examining a perfectly fair sample of 200 words it seems reasonable to assume that he would find among these 117 that he would know. And any conclusion we may draw from this hypothetical case will have a reliability of approximately $\sqrt{500} = 22.4$ times as much as it would have if it actually came from but a single trial. The fact that 200 is indeed a small sample will later appear in this, that the most probable mixture, even though it may be many times more probable than other mixtures not in the immediate neighborhood of the most probable one, is, as a matter of fact, a mixture whose probability is very small. This apparent paradox is often met with in problems involving large numbers.

A mere change in the wording of the problem makes the application of equation (2) evident. We have an urn filled with $b$ (= 104,000) words in unknown proportion of "white" (known) words to "black" (unknown) words. From this urn we draw 200 and find 117 white and 83 black. In other words the event in question occurs 200 times in one of two alternative ways, it occurring 117 times in the way which we may call successful. The probability that the dictionary contains a mixture of $x$ known and $b - x$ unknown words is then

$$P_x = \frac{\pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}}{\sum_1^n \pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}} \tag{4}$$

or

$$P_x = K\pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83}, K \text{ being a constant.} \tag{5}$$

We see at once that it is impossible to determine the value of $x$ for which this expression is a maximum without a knowledge of the character of the term $\pi_x$. This is exactly the point at which errors often creep into applications of the theorem. It is often assumed from the logical principle of insufficient reason that $\pi_x$ is a constant: in other words since we know too little to form a judgment it is assumed that the *a priori* probabilities of all causes are equal. The principle of insufficient reason leads, however, to notoriously paradoxical results.

For the problem here considered, however, while the theoretically correct result cannot be obtained without a knowledge of $\pi_x$, we can easily show that for

practical purposes our knowledge concerning it may be very limited and still sufficient.

Consider the curve

$$P'_x = \left[\frac{x}{b}\right]^{117}\left[\frac{b-x}{b}\right]^{83} \tag{6}$$

where $b = 104{,}000$. This curve has its maximum value when

$$x = (117/200)\, b = 61{,}000 \tag{7}$$

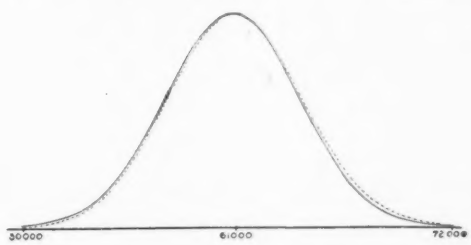
and the maximum value is equal to

$$P'_{61000} = 1.137 \times 10^{-59}. \tag{8}$$

However for $x = 25{,}000$

$$P'_{25000} = 2.207 \times 10^{-83}.$$

To obtain the value of $x$ for which $P_x$ of equation (5) is a maximum we have to multiply every ordinate of the curve $P_x'$ by $\pi_x$ (the existence probability of a mixture of $x$ known and $b - x$ unknown words), and then take the value of $x$ corresponding to the highest point on this new curve. Therefore unless the *a priori* existence probability of a mixture containing 25,000 known words exceeds the *a priori* existence probability of a mixture containing 61,000 known words in the ratio of $0.5 \times 10^{24}$ to 1 it is evident that the *a posteriori* probability of a mixture characterized by $x = 61{,}000$ is greater than the *a posteriori* probability of a mixture in which $x$ is only 25,000. In fact we have



$$\frac{P_{61000}}{P_{25000}} = \frac{K\pi_{61000} \times 1.137 \times 10^{-59}}{K\pi_{25000} \times 2.207 \times 10^{-83}} = 0.5 \times 10^{24}\frac{\pi_{61000}}{\pi_{25000}}, \tag{10}$$

which is greater than unity as long as

$$\pi_{25000} < 0.5 \times 10^{24} \times \pi_{61000}. \tag{11}$$

While this may convince us that the previous estimate of 25,000 is to be discarded it may not convince us that an estimate of, say, 50,000 is not as good a new estimate as that of 61,000 which the test indicates. Let us therefore consider the numerical magnitude of the ratio of the *a posteriori* probability of a mixture for which $x = 61{,}000$ to the *a posteriori* probability of a mixture for which $x = 50{,}000$. We have

$$\frac{P_{61000}}{P_{50000}} = \frac{\pi_{61000}}{\pi_{50000}}\left[\frac{61}{50}\right]^{117}\left[\frac{43}{54}\right]^{83} = 77\frac{\pi_{61000}}{\pi_{50000}},$$

which is greater than unity unless the *a priori* probability of a mixture of 50,000 known words exceeded the *a priori* probability of a mixture of 61,000 known words by the factor 77. Since neither of these figures, 50,000 and 61,000, is in any way special before the test is made there would seem no justifiable basis for considering one more probable than the other in any such ratio as that just found. The answer to our problem would then be that 61,000 is the most probable answer for the number of words in the dictionary, this conclusion being reached regardless of the character of $\pi_x$ outside of the one restriction stated in (11). It is understood, of course, that the character of the term $\pi_x$ in the neighborhood of $x = 61{,}000$ might shift the most probable value slightly, but $\pi_x$ would surely be changing very slowly for values of $x$ in this vicinity, and the shift would be therefore very small, and negligible for practical purposes. Although the question of whether equation (11) states a reasonable restriction upon the character of $\pi_x$ is primarily one for educators to settle it would certainly seem sensible to assume that it does. We should surely agree that the previous results were not sufficiently well established that we could consider them, *a priori*, $0.5 \times 10^{24}$ times as likely to be true as any other result. We must say "any other" result since our estimate of the *a priori* probabilities, it being independent of the result of the test and therefore for psychological reasons best formed before the test takes place, could attach no special importance to the figure 61,000—a number which is not known until after the test is performed. All we could say might be, for example, that we consider a result lying between, say, 20,000 and 30,000 one hundred times more likely than a result lying outside this band, and that we consider it certain that the actual mixture contains more than 5,000 and less than 90,000 words that the average student knows. Such an assumption, coupled with the fact that the total area under the curve $y = \pi_x$ must be unity gives

$$\left.\begin{array}{ll} \pi_x = 0 & 0 \leq x < 5000 \\ = 9.302 \times 10^{-7} & 5000 \leq x < 20000 \\ = 9.302 \times 10^{-6} & 20000 \leq x < 30000 \\ = 9.302 \times 10^{-7} & 30000 \leq x < 90000 \\ = 0 & 90000 \leq x \leq 104000 \end{array}\right\} \qquad (12)$$

Then we have $P_x$ given by the full line on the graph.

The vertical scale is again obtained from the fact that the area must equal unity. The probability of the most probable mixture is found to be $9.946 \times 10^{-5}$, a very small probability as was earlier suggested would be the case. The values of $P_x$ outside the range shown are too small to be indicated.

It is to be especially noted that this curve will approximately represent $P_x$ *whatever the assumption concerning* $\pi_x$, only provided, say, that

$$\pi_{25000} < .5 \times 10^{20}\, \pi_{61000} \qquad (13)$$

This condition is slightly more stringent than (11), and insures that $P_{25000}$ shall

be less than one ten thousandth of $P_{61000}$, and therefore negligible. It is easily shown that in a small neighborhood of $x = (117/200)\ b$ the equation

$$P_x = K\pi_x \left[\frac{x}{b}\right]^{117} \left[\frac{b-x}{b}\right]^{83} \tag{5}$$

is equivalent to

$$P_x = K\pi_x P'_{61000}\, e^{[-(117+83)3/2][(1/83)(1/117)]\epsilon^2}, \tag{14}$$

where

$$\epsilon = (117/200)b - x. \tag{15}$$

This equation reduces to

$$P_x = Ce^{-411.9\epsilon^2} \tag{16}$$

if we assume, as we have done, that $\pi_x$ is constant in a small range about $x = 61{,}000$. The constant $C$ is evaluated by means of the fact that when $\epsilon = 0$, $P_x$ must be the probability of a mixture containing 61,000 known words, which probability we have previously calculated. Then

$$C = 9.946 \times 10^5. \tag{17}$$

The size of the coefficient of $\epsilon^2$ in equation (16) indicates clearly how rapidly probabilities diminish in the neighborhood of the most probable result. The approximation of equation (16) to equation (5) is shown on the graph. The dotted line is the graph of equation (16).

We know from Bernoulli's theorem that if the dictionary actually consisted of $x^*$ words which we could characterize as known, and $b - x^*$ which would accordingly be unknown, as we take more and more samples from it the estimate formed from these samples must approach the value $x^*$. We have, indeed, for the ratio of the *a posteriori* probability of a mixture containing $x$ known words to the *a posteriori* probability of a mixture containing 61,000 known words

$$\frac{P_x}{P_{61000}} = \frac{\pi_x}{\pi_{61000}} \left[\frac{x}{61000}\right]^m \left[\frac{104000 - x}{43000}\right]^{k-m},$$

where $m$ known words have appeared in a total sample of $k$. If $m/k - m$ equals 61,000/104,000 the above ratio has its maximum when $x = 61{,}000$, in which case it is obviously unity. For any other value of $x$ this ratio may be greater than unity for a given $k$, but must become and remain smaller than unity, as $k$ increases indefinitely whatever the (finite) ratio of $\pi_x$ to $\pi_{61000}$. The truth of this statement is obvious from the form of the above equation. It thus appears that the *a priori* probability is vanishingly unimportant as the number of trials increases. If it were the case that 500 tests had been performed with the invariable result of 117 known out of 200 chosen in each test we would have $k = 100{,}000$ and $m = 58{,}500$. It is then clear that the above ratio would be exceedingly small for any value of $x$ other than 61,000 practically independently of the ratio $\pi_x/\pi_{61000}$.

It is, however, not strictly admissible to make such a calculation in our case. For one thing the result in the 500 tests differed, even though with surprisingly

small variability. Even more important than this, however, is the fact that we cannot in all strictness compare our problem with the analogous urn problem in case the test is used on more than one person, as it of course was. For the content of the dictionary, from our point of view, actually changes with the observer, depending, as it does, upon how many words *he* knows. This is an added source of variability, over and beyond that which would occur due to the ordinary errors of sampling. The result of the above paragraph is still qualitatively applicable.

Our final conclusion is, then, that the experimental evidence of the test completely justifies us in abandoning the old result and accepting the new: and that, moreover, probabilities of mixtures in the immediate neighborhood of the most probable mixture themselves follow the normal Gaussian law as given by equation (16).

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## A GRAPHICAL AID IN THE STUDY OF FUNCTIONS OF A COMPLEX VARIABLE.

By NORMAN MILLER, Queen's University.

The impossibility in three dimensions of representing graphically a function of a complex variable makes it necessary for the student to call on his imagination in other ways in order to realize the properties of these functions. Two methods are common in the geometrical theory of functions. One is to represent in two different planes or in two Riemann surfaces the variables $z$ and $w$ and to study the correspondence between the points of the two planes or surfaces, which is determined by the relation $w = f(z)$. The second method, which does much to illuminate the subject for the beginner, is to represent in one plane both the independent and dependent variables and to interpret the transformation kinematically as a flow of the points in the plane.[1]

A complete graph of the function $w = f(z)$ or $u + iv = f(x + iy)$ consists of a 2-dimensional manifold in space of four dimensions. Nevertheless the student, in his effort to visualize the function, thinks instinctively of a surface spread out over the plane of $z$. Such a surface is actually determined by taking for a third coördinate the absolute value of $f(z)$. Calling the third coördinate $\zeta$ the equation of the surface is

$$\zeta = \sqrt{[u(x, y)]^2 + [v(x, y)]^2},$$

only the positive square root being taken. In this representation all points on a circle of center 0 in the $w$-plane yield the same ordinate $\zeta$. It is, in fact, by making no distinction among the points of such a circle that we are able to pass from a two-way spread in four dimensions to an actual surface in three dimensions.

It is interesting to enquire what properties of the function $f(z)$ are exhibited

[1] See in this connection an article by Cole, *Annals of Mathematics*, vol. 5, June, 1890.

in the surface $\zeta = |f(z)|$ and what properties are lost. The following discussion will apply only to functions which are in general analytic. At a zero of such a function the surface reaches down to the plane of $z$ and in the neighborhood of a pole it extends upward indefinitely. It will easily be seen that the following properties are among those put into evidence in the surface.

(1) A function which is analytic over the entire plane and remains finite is a constant. (Liouville.)

If $f(z)$ is analytic in the plane without exception and the surface $\zeta = |f(z)|$ does not rise indefinitely then it can only be a plane parallel to the $z$-plane.

(2) The zeros of an analytic function which is not identically zero are isolated points.

The surface cannot touch the $z$-plane along a curve unless it is the $z$-plane itself. A zero may give rise to a conical point on the surface, it may belong to an edge, or it may be an ordinary point at which the $z$-plane is tangent.

(3) The poles of a function which is otherwise analytic are isolated points.

If from each pole a perpendicular be erected to the plane of $z$ the surface rises indefinitely about each of these lines and nowhere else.

(4) Let $f(z)$ be any function other than a constant, analytic except for poles within and on the boundary of a region $S$. Then, excluding the zeros and poles, $|f(z)|$ can have neither a minimum nor a maximum at any other interior point of $S$. The same property may be stated more precisely as follows. In any subregion of $S$ which includes no poles either within or on its boundary $|f(z)|$ takes on its maximum value on the boundary and in any subregion which includes no zeros $|f(z)|$ takes on its minimum value on the boundary.

If, then, a cylinder be erected on any closed curve $C$ of $S$, with elements perpendicular to the $z$-plane it will cut from the surface a portion which will have its highest point on the boundary provided $C$ includes no poles and which will have its lowest point on the boundary provided $C$ includes no zeros.

(5) In the neighborhood of an isolated essentially singular point the function comes arbitrarily near to every assigned value (Weierstrass). It follows that $|f(z)|$ comes arbitrarily near to every positive real value. If a line be drawn upward from the singular point perpendicular to the plane of $z$ then a point moving in the surface can approach as closely as we like every point on this line. The effort to visualize the surface in the neighborhood of an essentially singular point brings out clearly the nature of the singularity.

Models of the surfaces $\zeta = |f(z)|$ for some of the simpler functions would be of value in teaching the elementary theory of functions. Evidently $\zeta = |z|$ is one sheet of a circular cone with vertex 0 and axis of $\zeta$ for axis. $\zeta = |z^2|$ is a paraboloid of revolution with vertex touching the $z$-plane at the origin. The character of the surfaces for some other simple functions is shown in the figures 1–5. Only a portion of each surface is drawn.

(1) $\zeta = |z(z-1)|$ touches the $z$-plane at the points 0 and 1. Let us obtain the Cartesian equation of this surface.

$$z(z-1) = (x+iy)(x+iy-1)$$

$$= x^2 - y^2 - x + i(2xy - y),$$

$$\zeta^2 = (x^2 - y^2 - x)^2 + (2xy - y)^2.^1$$

Translating the origin to the point $(\frac{1}{2}, 0, 0)$

$$\zeta^2 = (x^2 + y^2)^2 - \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{16}.$$

The section of the surface by a plane $\zeta = k$ is a Cassinian oval which becomes for $k = \frac{1}{4}$ the familiar lemniscate.

(2) $\zeta = |e^z| = e^x$. The surface is a cylinder whose elements are parallel to the axis of $y$ and whose normal section is the exponential curve.

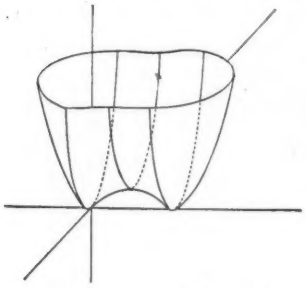

FIG. 1. $\zeta = |z(z-1)|$.

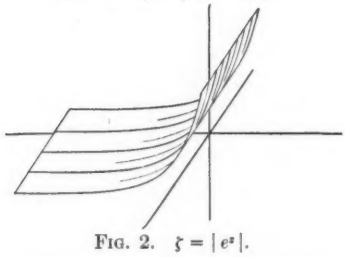

FIG. 2. $\zeta = |e^z|$.

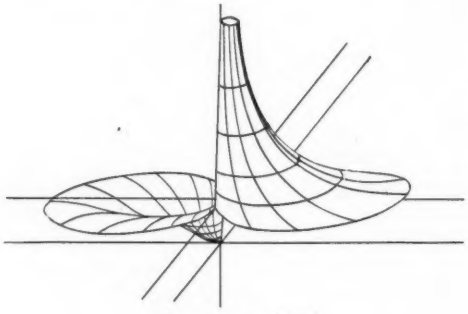

FIG. 3. $\zeta = |e^{1/z}|$.

(3) $\zeta = |e^{1/z}|$. This surface is obtained from the former one by making an inversion in the unit cylinder about the axis of $\zeta$.[2] It consists of two parts,

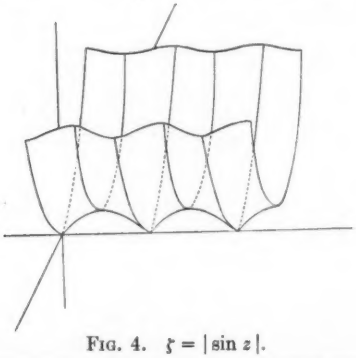

FIG. 4. $\zeta = |\sin z|$.

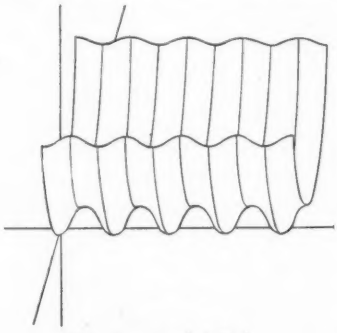

FIG. 5. $\zeta = |\sin^2 z|$.

[1] This equation denotes 2 symmetrical surfaces of which we are concerned only with the one above the $z$-plane.

[2] The surface being symmetrical in the $x\zeta$-plane a reflection in this plane does not alter the character of the surface. It does of course alter the correspondence of points.

one above the plane $\zeta = 1$ and to the right of the plane $x = 0$, the other below the plane $\zeta = 1$ and to the left of $x = 0$. Each sheet of the surface approaches the plane $\zeta = 1$ asymptotically.[1] Every section of the surface by a plane parallel to the $z$-plane is a circle. The nature of the essential singularity at $z = 0$ is well illustrated.

(4) $\zeta = |\sin z|$ and (5) $\zeta = |\sin^2 z|$. The Cartesian equations of these surfaces are easily found to be $\zeta = \sqrt{\sin^2 x + \sinh^2 y}$ and $\zeta = \sin^2 x + \sinh^2 y$. The surfaces show the periodic character of the functions and the fact that $|\sin z|$ may take on any positive value in a period strip.

These examples are intended only to illustrate and not to exhaust the possibilities of the subject. The student should try to visualize the surface $\zeta = |\sin 1/z|$ in the neighborhood of its essential singularity. A few hours spent in constructing these surfaces will contribute towards overcoming the feeling of "a stranger in a strange land" which most students experience in entering upon the study of the theory of functions of a complex variable.

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## MODULAR GEOMETRY.

By ALBERT A. BENNETT, Baltimore, Md.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The study of relations in a single variable, in the case of finite number fields, is not, of course, a new idea, although it is awakening perhaps more interest now than in the last century. A study of several variables and in particular a characteristically geometrical treatment for finite fields may fairly be said to have been introduced within the past few years, and the only extended discussions of the whole subject to be found in any general treatise are those in Veblen and Young's *Projective Geometry*, and in a little book by G. Arnoux, *Essai de Géométrie Analytique Modulaire*, Paris, 1911. References are given in L. E. Dickson, *On Invariants and the Theory of Numbers* (Madison Colloquium Lectures), 1914, page 98.

### § 1. Plane Geometry, Modulo 3.

We shall for the sake of concreteness, consider the individual case of the field, modulo 3. This may be thought of as generated by the set of residues obtained by dividing integers by the number 3. Thus any integer, $n$, falls in one of three mutually exclusive classes according as $n$, $n - 1$, or $n - 2$ is an integral multiple of 3. We say that $n$ is respectively congruent to 0, 1 or 2, expressed in symbols by $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$, respectively. There are then only three numbers, 0, 1, 2, in the field. For these we have the following addition and multiplication tables.

[1] Except in the $y\zeta$-plane. The plane $\zeta = 1$ intersects the surface in a line parallel to the $y$-axis.—Editor.

Addition:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

Multiplication:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

Subtraction and division (except by 0) are always uniquely possible. Incidentally $-1 = \frac{1}{2} = 2$, in this field. The equation $x(x-1)(x-2) = 0$, has 0, 1, 2, as roots, and therefore $x^3 - x = 0$ is identically satisfied by every number of the field. Every linear equation with coefficients in the field has a unique root in the field, but quadratic equations exist with coefficients in the field whose roots are imaginary with respect to the field, that is, are not themselves in the field but may be introduced by a consistent extension of the system. Thus $x^2 + 1 = 0$, is not satisfied by 0, 1, or 2, and hence has no real roots in the field. Every polynomial with coefficients in the field may be successively reduced if of degree higher than 2 by means of the identity $x^3 = x$, so as to be represented as of degree not greater than 2. The following are therefore the totality of essentially distinct polynomials in the field:

$$0,\quad 1,\quad 2,\quad x^2,\quad x^2+1,\quad x^2+2,\quad 2x^2,\quad 2x^2+1,\quad 2x^2+2,$$

$$x,\quad x+1,\quad x+2,\quad x^2+x,\quad x^2+x+1,\quad x^2+x+2,\quad 2x^2+x,\quad 2x^2+x+1,\quad 2x^2+x+2,$$

$$2x,\quad 2x+1,\quad 2x+2,\quad x^2+2x,\quad x^2+2x+1,\quad x^2+2x+2,\quad 2x^2+2x,\quad 2x^2+2x+1,\quad 2x^2+2x+2.$$

One may proceed in this manner to study functions of one variable. But our present interest is geometry, taken for simplicity as of two dimensions, so that we shall not inquire further into properties of polynomials in one variable.

By points in the non-homogeneous modular plane (*mod* 3) will be meant any set of objects each of which is regarded as determining uniquely and being uniquely determined by an ordered pair of numbers of the modular field. The number of distinct points is equal to the number of distinct ordered pairs, $(m, n)$, where each of these is in the field. The plane consists therefore of nine points. By a straight line in the plane is meant the set of points satisfying a single linear equation in two unknowns, the whole being in the field. Each line is found to contain three points. The set of distinct lines corresponds to the following, and are twelve in number.

$$x = 0,\quad y = 0,\quad x + y = 0,\quad x + 2y = 0,$$

$$x = 1,\quad y = 1,\quad x + y = 1,\quad x + 2y = 1,$$

$$x = 2,\quad y = 2,\quad x + y = 2,\quad x + 2y = 2.$$

The equation $2x + y = 1$, for example, is identical in the field with $x + 2y = 2$, either equation being obtained from the other by multiplying by 2. Through any point there will be four distinct lines of the plane. Lines are either parallel or intersect in a point of the plane as in the usual system. For example,

$2x + y = 1$, and $x + y = 2$, intersect in a point whose coördinates may be obtained by the usual method of determinants, viz.,

$$\left(\frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}\right) = (2, 0).$$

Quadratic loci may be studied in the usual manner. In particular, there are circles. These are of three classes according as the square of the radius is 0, 1 or 2. A circle with imaginary radius, *e.g.*, $\sqrt{2} = \sqrt{-1}$, has in this geometry real points. For example, $x^2 + y^2 = 2$, contains the points (1, 2), (2, 1), (1, 1), (2, 2), which are as numerous as the points on the circle, $x^2 + y^2 = 1$, which has a real radius, namely the points, (1, 0), (2, 0), (0, 1), (0, 2). The circle of zero radius contains as usual but one point. Thus the three circles with any given common center, together contain all the points of the plane, or in other words a circle may be drawn with a given center and passing through a given point. It is interesting to notice that the circle of radius one is also of radius two, since in this system $2 = -1$.

The trigonometry in this plane is rather primitive since the only angles occurring are multiples of 45 degrees. The trigonometric functions yield the usual table reduced modulo 3.

| | sin | cos | tan | cot | sec | csc |
|---|---|---|---|---|---|---|
| 0° | 0 | 1 | 0 | – | 1 | – |
| 45° | $2i$ | $2i$ | 1 | 1 | $i$ | $i$ |
| 90° | 1 | 0 | – | 0 | – | 1 |
| 135° | $2i$ | $i$ | 2 | 2 | $2i$ | $i$ |
| 180° | 0 | 2 | 0 | – | 2 | – |
| 225° | $i$ | $i$ | 1 | 1 | $2i$ | $2i$ |
| 270° | 2 | 0 | – | 0 | – | 2 |
| 315° | $i$ | $2i$ | 2 | 2 | $i$ | $2i$ |

where however $i$ is not a real number but may be introduced as an imaginary quantity satisfying the relation $i^2 = 2 = -1$.

Ellipses, parabolas, hyperbolas, all exist with their usual properties although many of these become trivial. Higher plane curves do not exist in the real plane.

As in the ordinary geometry, real points fall into three classes with respect to a non-degenerate circle: those points from which there are two real distinct tangents,[1] those with but one real tangent, those with no real tangents to the circle. Ordinarily the first is the class of exterior points, the second, the class of points on the circle, and the third the class of interior points of the circle.

[1] By tangent we mean the line whose equation is formed as in the ordinary analytical geometry. For example, the tangent to the circle $x^2 + y^2 = 1$ at a point $x'$, $y'$ will be the line whose equation is $xx' + yy' = 1$.

Were this adopted as a definition we would say that for every non-degenerate circle in this modular plane there is one interior point, the center, four real points on the circle and four exterior points. Unfortunately for intuition, the four points exterior by this definition to a circle, $C_1$, are on a circle, $C_2$, with the same center, while the four points on $C_1$ are by this definition themselves exterior to $C_2$. The line joining an exterior point with the center of the circle does not meet the circle in real points.

## § 2. Imaginaries in Modular Geometries.

Whereas for ordinary plane geometry the introduction of all imaginaries may be effected in one step, this is not true in the same sense for modular geometries. This is a consequence of the failure in modular geometries of the following theorem, valid for the numbers of ordinary analysis: While an algebraic equation with real coefficients may fail to have a real root, the system of complex numbers, $a + ib$, where $a$ and $b$ are real and $i$ is a square root of minus one, is such that every algebraic equation with complex coefficients (of degree greater than zero), has a complex root (real numbers being included in the term, complex).

This is of course the so-called fundamental theorem of algebra. It must be distinguished from the sub-case frequently employed which states that every algebraic equation with real coefficients (of degree greater than zero) has a root which is expressible in the form $a + ib$.

For a modular geometry a theorem analogous to this sub-theorem is still true, but the general theorem first stated must be much extended. Thus if in a modular geometry an algebraic equation with real coefficients be given, it is always possible to introduce a single "imaginary" quantity, $i$, such that the given equation has a root of the form $a + ib$, where $a$ and $b$ are real. Unfortunately the $i$ depends upon the degree of the original equation, assumed irreducible, and a new equation with coefficients involving $i$ will require a further extension of the system. The system requires an infinite number of steps of this sort before it is complete.

We have mentioned only the Euclidean analogue, but the classical non-Euclidean geometries and the more general projective geometry have also corresponding modular geometries.

## § 3. Significance of Modular Geometry.

The study of modular geometry is justified from several points of view. In the first place, an investigator drilled in the concepts of general algebraic geometry is better capable of distinguishing in a given problem the essential from the non-essential features, and will seek recourse in complicated relations only when these are inherent in the problem. By avoiding extraneous machinery, proofs are secured which are at the same time economical and of wide applicability.[1]

[1] The points of a given Euclidean space being assumed, and the notion of dimensionality being established, one may distinguish among the remaining geometric notions two categories having very little beyond the concepts of point and dimension in common. These may be roughly identified with the terms "continuity" and "algebraic rationality." In the one case there is a body

In the second place finite groups, in particular multiply transitive groups can regularly be interpreted and frequently with profit, in the geometrical language of a suitably chosen modular space. The abstract group becomes in many cases geometrically intuitive only by reference to an interpretation of this sort. In the third place the subject still presents novelty and has interest on its own account, the methods of investigation are extremely simple, and most of the results for a given modular space are obtained with ease and completeness.

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## QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

### REPLIES.

**15** [*1914*, 278; *1916*, 353; *1920*, 114]. In the *Proceedings of the Royal Society of Edinburgh*, Vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

### REMARKS BY THE EDITOR.

A number of communications relative to this question have been received. As regards content they fall into three classes. In the first (and largest) class may be placed mere demonstrations of the algebraic premiss of the question,—that if $x^3 + y^3 = z^3$, then

$$(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3.$$

This is indeed easy, and we are not to suppose that the question implied any difficulty in obtaining an identity within the reach of any high school student who has learned the fundamentals of algebraic notation. It seems necessary to emphasize, therefore, that the truth of the algebraic identity is granted, and that what is desired is an "easy" proof, based on this identity, that the equation $x^3 + y^3 = z^3$ has no solutions in positive integers.

In a second group fall the replies indicating that Tait may have been mistaken. One suggestion is that he may have noted that any of the relations $x = y$, $x = -z$, $y = -z$ will reduce the second equation to an identity, but not the first. This is not impossible; it amounts to supposing that Tait considered the two equations as equivalent,—that he regarded the second as being not only a necessary, but also a sufficient condition for the first. If so, he would not be the only mathematician of note who has confused necessary and sufficient conditions; but such an error would be so obvious in the present instance as to seem rather unlikely. Another correspondent suggests that Tait took for granted that an equation of the form $x^n + y^n = z^n$ cannot hold for an infinite number of sets of values of $x$, $y$, $z$ when $n > 1$. Of course, however, this

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of definitions and theorems concerned only with continuity and forming a science which may be called the Analysis Situs of the given space; in the other case there is a corresponding science treating of algebraic relations among figures describable in geometrico-algebraic language.

By employing the term "algebra" in a wide sense to apply to commutative number fields (that is, number fields with commutative multiplication), one may study a "general algebraic geometry," which will include as sub-cases all those distinct instances that arise by further characterizing the commutative number field under consideration. The entire science may be regarded as the study, in geometrical language to be sure, of algebraic relations in a given number of variables defined over a general commutative number field. Theorems in this general algebraic geometry are clearly independent of Analysis Situs since, as has been shown, number fields may be selected in which continuity has no application.

relation *does* hold for an infinite number of cases when $n = 2$; and we can scarcely ascribe to Tait such ignorance of the status of Fermat's theorem as this assumption would imply.

The third class of replies consists of those which seek to show that from a solution of $x^3 + y^3 = z^3$ can be deduced another whose elements are smaller, and hence to apply the method of infinite descent. This is the most reasonable direction in which to look for Tait's "easy" proof. Unfortunately all proofs of this type which have been sent in are fallacious.

**30** [*1916*, 88, 354; *1920*, 114]. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

## Reply by H. E. Webb, Central High School, Newark, N. J.

An answer to this question is conditioned upon the state of mind of the high-school teachers in question. If they have never taken college mathematics at all, thirty-five hours is as a teacupful of water to a thirsty horse. It should be borne in mind that high-school students often have capacity for reasoning far beyond that with which they are credited; but at the same time that they object to various canonical forms.

A teacher of high-school algebra should have a knowledge of the fundamental principles of number such as are found in the early pages of any good text in the elementary theory of functions of the real variable. He should have also some *clear* notions regarding series, equations generally, and the significance of a complex number system. He will then find that many of the difficulties which his students encounter are due to a dimly realized sense of the need of distinctions which he himself is able to draw clearly. He should also be familiar with the elements of the analytic geometry of the straight line and the conics, in order properly to weigh the value of various topics and examples for the future practical or theoretical development of mathematics. He is presumably familiar with plane trigonometry, as this is now accounted a high school subject.

A teacher of elementary geometry should above all else be familiar with the fundamentals of synthetic projective geometry, together with the logical foundations of metrical Euclidean geometry, in order to escape the embarrassment of finding that his pupils' logic is better than his own.

To bring these various topics into a compass of thirty-five hours calls for a high degree of skill on the part of the instructor. But it may be noted that no great amount of drill in technique is needed, and that many sub-topics which are often presented in college courses may be omitted from an intensive course of this sort.

The college course which seems least of all to bear directly upon secondary mathematics is the calculus, excepting in so far as the derivative may be used in analyzing higher equations. But this seems to many out of place in the high school.

**37** [*1919*, 151; *1920*, 115]. Criticize the following as fundamental definitions of elementary geometry:

A *plane surface* is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A *straight line* is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

## I. Reply by A. A. Bennett, Baltimore, Md.

The above will be criticized from three aspects,—logical, postulational, and psychological.

From a logical standpoint, the above "definitions" are faulty in that they fail to define. While it is true that a sphere or circle may be viewed abstractly to the extent that the location of the center with respect to other objects need not be specified, caution must be observed when limits are to be used. The limit of a "finite portion" is not definite. As the radius is increased, the finite portion must change, but *how*? If the center remain fixed, a plane or line obtained in accordance with directions could contain no finitely accessible point. If the center move, why might not the subtended angle vary also? By steadily diminishing the angle, the resulting limit might be bounded or even reduce to a point. Unless care be exercised there will be no limit at all. Logical objections (which are numerous) can be met, however, by sufficient care in statement.

Geometry cannot afford to resemble a circle in having neither beginning nor end; nor ought its center of interest be removed "to infinity." Somewhere terms must be introduced for the first time which shall become fundamental for all further developments. While a geometry of inversion may properly start with spheres and circles, Euclidean geometry is usually found to be more tractable when starting with lines and planes. The above "definitions" imply a reversal of practice and leave one wondering as to what shall be done with spheres and circles. Any discussion involving isolated definitions implies a background of a system in which these are to find their place. No extended discussion of the proposed definitions is appropriate until at least a sketch of the development of Euclidean geometry from spheres and circles, the notion of radius, applications of limits, etc., all prior to the mention of lines and planes, is vouchsafed.

Psychologically, curved surfaces and lines appear more complicated than linear ones. Any use of the notion of limits is at least unfortunate in definitions, until the existential character of the object seems to demand it. In the definition of the length of an arc, area of a surface, etc., limits are usually necessary and appropriate. The "incommensurable case" depending upon the existence of irrational numbers, is a different but analogous instance in which limits are sometimes at least appropriate. Even where limits are freely employed few applications would be more objectionable in an elementary geometry on a non-projective basis than the discussion of what happens when a point moves off "to infinity," a region for which the Euclidean geometer has an abhorrence. The circle as the limit of polygons suggests merely an abstraction which can be visualized in a finite portion of space, and even lines were formerly avoided in favor of finite segments.

A careful introduction in a course supplementary to the usual elementary geometry, of the notion of points at infinity, coincident points, tangents as the limit of secants, general and degenerate cases, and incidentally of lines as special members in linear systems of circles—but *not* for the purpose of definition—would surely give the student valuable breadth of vision.

## II. Reply by H. E. Webb, Central High School, Newark, N. J.

These definitions are open to many serious objections: In the second definition, the expression "the radius increases" can be interpreted only by employing the straight line concept itself. A beginner is not slow to notice this. The definition also violates the usual assumptions of order for points on a line in Euclidean geometry, and renders impossible many familiar demonstrations. A necessary consequence of the definition would be the equality of radii of two straight lines, since otherwise there would be no laws of congruence in the Euclidean sense. If, however, straight lines are assumed to have equal radii, then any two of them must intersect, and we have an elliptic geometry, which is interesting in its way, but is not Euclidean. From an elementary standpoint it is undesirable to define one configuration as the "limit" of another. Much confusion has in the past resulted from the misuse of the term "limit" in elementary texts. A "limit" should be regarded as a number, and a variable as any one of a class of numbers, all of which are determined according to some law. The idea of a Dedekind cut, explained in simple terms and not too nicely, is easily understood by students who have had two years of elementary mathematics. The numerical measure of the circumference of a circle can be explained in this way. But at the outset of the study of geometry the attempt to define a straight line as the limit of anything else would make a certain unavoidable confusion worse confounded.

Attempts in elementary geometry to define a *straight line* are usually worse than useless.

The first definition could presumably be so reshaped as to be correct, as far as phraseology is concerned, but would be open to the other objections mentioned above; it would also necessitate a restatement of most of the familiar definitions, and would serve no useful purpose as compared with the usually accepted criterion for points on a plane.

## III. Remarks by the Editor.

It does not seem certain that the second definition necessitates equal "radii" of straight lines or leads to an elliptic geometry, as indicated in Mr. Webb's reply. In fact, the definition is so vague as to permit a variety of conjectures as to the interpretation which must be made in order to draw conclusions from it. Both the preceding replies exhibit beyond doubt the utterly unsuitable character of the proposed definitions for inclusion in a logical basis of elementary geometry. With respect to the last sentence of Professor Bennett's reply the Editor would express his belief

that use of the term "coincident points" should be permitted, if at all, only to the adept; never to the disciple, who may so easily, in this as in other similar cases, mistake an analogy for a proof.

**38** [*1919*, 151; *1920*, 115]. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

## I. Reply by H. E. Webb, Central High School, Newark, N. J.

This question is very well opened for discussion at this time. It is safe to say that in the teaching of Euclidean plane and solid geometry in secondary schools and colleges there has been an over emphasis upon finer points of logic at certain stages, together with the swallowing of a host of mathematical camels at others. This, more than anything else, has turned students against the subject. Nearly every student has some sense of logical accuracy, and he is, to say the least, annoyed to find it outraged at one moment by what seems a serious break and insulted at the next by a captious insistence upon detail. The difficulty is reducible by careful analysis, and it is earnestly to be hoped that this will be undertaken soon. In the writer's humble opinion the solution, as far as Euclidean geometry is concerned, consists in establishing an arbitrary (or conventional) distinction between projective geometry and metrical geometry, and in the assembling of all of those theorems of projective geometry which are useful for metrical geometry as a list of assumptions with which to begin the latter, in total disregard of any desire to keep this list at a numerical minimum, and without too great fear of redundancy.

In the field of algebra and elementary analysis there is less confusion of the sort referred to, for the simple reason that even the ordinary demands of logic are so often thrown to the winds in favor of a parrot-like repetition of meaningless phrases. The consequent logical absurdities are gracefully "side-stepped" by the instructor. It is a fact that in the development of a subject not all the necessary steps can always be taken. But when such steps are omitted, the omission should be plainly marked. In other words, the instructor may well take a leaf from the book of the student who, upon encountering a particularly difficult demonstration, reads enough of it to obtain a fair idea of its meaning and then passes it by in favor of something which in his judgment is more likely to appear upon the final examination. In doing so he shows an excellent sense of discrimination.

The occasion of the disparity in proportion of time spent upon logical development, on the one hand, and upon drill in technique, on the other, is of course the lack of time for a comprehensive treatment of any of the mathematical branches in the required courses in school and college. This condition is not likely to be corrected immediately. In the field of algebra certainly, and probably also in that of the elementary calculus, emphasis upon technique is the natural result of a mistaken effort to elevate standards of instruction through the medium of written examinations; these should be difficult enough that the student who is bright but lazy cannot "cram" through them. As a result the student who is diligent but less gifted often has a very hard time of it, and the instructor, in a laudable effort to aid and encourage him, loses sight of essential principles, in the organizing of technique. This state of affairs is obviously less easily attained in those branches which lay greater emphasis upon synthesis than upon analysis.

A valuable aid to teachers would be a comparative analysis of relative difficulty of problems in some limited field, based upon the number of distinct principles involved in each problem.

Mathematical principles, which are likely to be remembered, are of much greater worth to teach than tricks of the mathematical trade, which are soon forgotten.

## II. Reply by T. G. Rodgers, New Mexico Normal University.

A discussion of this question requires a large experience in both the secondary and collegiate fields of mathematics. Granted this experience, few can speak with equal assurance in both fields. Hence this discussion is confined to the secondary field.

The tendency to stress technique in algebra and logic in geometry seems to grow out of three sources:

1. It is generally believed that technique is more easily acquired by the immature student than logic, and traditionally algebra comes first in the high-school course;

2. Geometric figures form a more concrete basis for logic than algebraic symbols;

3. Elementary geometry has been bequeathed to this modern world as a monument of rigorous logic which too many feel it would be profane to treat in any other way.

From experience, observation and experiment I am convinced that the above tendency, whatever its cause, violates sound principles of teaching. Algebra and geometry should be taught side by side in the high school, not one before the other, and technique and logic should receive about equal stress throughout the course. Technique without logic and logic without action are alike foreign to the adolescent. It must, however, be borne strictly in mind that insistence upon difficult technique causes as many failures as insistence upon rigorous logic; both must be kept within the ability of the high-school student. In a word, the development of technique and logic should receive equal attention; the material for the development of technique should be simplified and enriched by the introduction of geometrical drawing and construction work; the logic sought should be the logic of the high-school student, not that of the expert logician; technique should gradually increase in fineness and logic in rigor as the course advances.

This procedure will aid to develop the power to understand the quantitative relations of the everyday world that confront the student and the ability to put this understanding into successful execution. The procedure of question 38 has a tendency to develop memorists in logic and jugglers in technique.

## NEW QUESTIONS.

**40.** How great emphasis is laid in freshman mathematics upon the elementary algebra of complex numbers? A recent paper by an eminent electrical engineer seems to indicate the need of a knowledge of this subject on the part of draftsmen and mechanics of very limited educational opportunities. The syllabus of the College Entrance Examination Board mentions the topic under the caption "Advanced Algebra," but the question papers call for only the slightest study of numbers of this type. Is Euler's theorem $(e^{i\theta} = \cos\theta + i\sin\theta)$ usually presented in college algebra, or is it left to the calculus? Does the topic deserve greater emphasis than it usually gets, for the sake of applications in the field of periodic currents?

**41.** A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

*Every positive integer of the form* $8n + 3$ *is the sum of three odd squares.*

*Every positive integer is the sum of not more than three triangular numbers.*

NOTE. Bachmann[1] states that these theorems have been proved only by the use of the theory of ternary quadratic forms, and refers to the discovery of the theorems by Gauss, and a comparatively simple proof by Dirichlet,[2] by means of this theory.—EDITOR.

## DISCUSSIONS.

Professor Noble, discussing the familiar fact that the addition of two rational algebraic fractions each in its lowest terms, by the usual method of reducing to the least common denominator, may lead to a result not in its lowest terms, shows in what way this situation presents itself, and verifies the process suggested by one text-book for securing the result in reduced form. That a similar contingency may arise in arithmetic is clear to anyone who has performed the addition of $\frac{1}{2}$ and $\frac{1}{6}$.

Professor Johnson calls attention to a very simple formula for an approximation to the smaller acute angle of a right triangle in terms of the sides, in which the error is surprisingly small. The formula was given, as the author states,

[1] *Niedere Zahlentheorie*, Leipsic and Berlin, 1910, Teil 2, p. 325.

[2] *Crelle's Journal*, Vol. 40, p. 228; *Liouville's Journal*, Series 2, Vol. 4, p. 233.

by Ozanam,[1] but it is in fact much older, having been announced by Nicolaus Cusanus.[2] It has been discussed by various writers,[3] with regard to its proof and estimates of its accuracy; the references were not accessible to Professor Johnson. Although his proof is essentially the same as that in Mansion's first paper, cited in the footnote, it has seemed worth while to publish it, for the sake of calling attention to this remarkable approximation, which does not seem to be familiar to most mathematicians.

By equating to zero the derivative of the expression

$$\frac{172 \sin B}{2 + \cos B} - \frac{180B}{\pi}$$

we find that the maximum positive deviation occurs for $B = 22° 20'$ approximately, and is equal to about 0.0122° or less than 45″. The maximum negative deviation, as indicated by the author's tables, is 0.0730° or between 4′ and 5′.

## I. Note on the Solution of Fractional Equations.

By Charles A. Noble, University of California.

In Wilczynski and Slaught's *College Algebra* the following interesting statement occurs on page 242:

"If, however, we express every one of the rational fractions as a sum of simple partial fractions, then unite all of the partial fractions which have the same denominator into a single one, and finally add these partial fractions together, using as a common denominator the lowest common multiple of the denominators of the simple partial fractions, we may be sure that the sum obtained in this way is in its lowest terms."

The correctness of this rule was considered evident by the authors and the proof of it was omitted. The following is a formal proof:

Let $F$, $G$, $P$, and $Q$ be polynomials in $x$. Assume first that $G$ and $Q$ have only one common linear factor, and that each has only one other linear factor, where each factor may be present in any order of multiplicity. Let $F/G$ and $P/Q$, assumed to be in their lowest terms, be expressed in partial fractions:

$$\frac{F}{G} = \frac{A_1}{(x-\alpha)^k} + \frac{A_2}{(x-\alpha)^{k-1}} + \cdots + \frac{A_k}{x-\alpha} + \frac{B_1}{(x-\beta)^\lambda} + \frac{B_2}{(x-\beta)^{\lambda-1}} + \cdots + \frac{B_\lambda}{x-\beta},$$

$$\frac{P}{Q} = \frac{C_1}{(x-\beta)^\mu} + \frac{C_2}{(x-\beta)^{\mu-1}} + \cdots + \frac{C_\mu}{x-\beta} + \frac{D_1}{(x-\gamma)^\nu} + \frac{D_2}{(x-\gamma)^{\nu-1}} + \cdots + \frac{D_\nu}{x-\gamma}.$$

[1] *Nouvelle trigonométrie, où l'on trouve le moyen de calculer toutes sortes de triangles rectilignes sans les tables de sinus, . . .*, 1699.

[2] *De mathematica perfectione; Opera*, Paris, 1514.

[3] F. A. Protche, *Mémoires de la société académique de l'Aube*, Vol. 51 (1887), pp. 149–162.

H. Brocard, *Mathesis*, 1889, p. 161; P. Mansion, ibid., p. 162; P. Mansion, ibid., p. 181.

Further references on closely related formulas may be found in Th. Vahlen, *Geometrische Konstruktionen und Approximationen*, Leipsic, 1911, pp. 188–206.

Assume $\lambda \geqq \mu$ and add:

$$\frac{F}{G}+\frac{P}{Q}$$

$$= \frac{(x-\beta)^{\lambda}(x-\gamma)^{\nu}\{A_1+A_2(x-\alpha)+\cdots\}+(x-\alpha)^{k}(x-\gamma)^{\nu}\{B_1+B_2(x-\beta)+\cdots\}}{(x-\alpha)^{k}(x-\beta)^{\lambda}(x-\gamma)^{\nu}}$$

$$+ \frac{(x-\alpha)^{k}(x-\beta)^{\lambda-\mu}(x-\gamma)^{\nu}\{C_1+C_2(x-\beta)+\cdots\} + (x-\alpha)^{k}(x-\beta)^{\lambda}\{D_1+D_2(x-\gamma)+\cdots\}}{(x-\alpha)^{k}(x-\beta)^{\lambda}(x-\gamma)^{\nu}}.$$

Now the sum on the right, with the common denominator $(x-\alpha)^{k}(x-\beta)^{\lambda}(x-\gamma)^{\nu}$ will be in its lowest terms unless its numerator has $x-\alpha$, $x-\beta$, or $x-\gamma$ as a factor. $x-\alpha$ will be a factor if, and only if, $A_1 = 0$. But this implies that $F/G$ is not in its lowest terms. Similarly, $x-\gamma$ will be a factor if, and only if, $D_1 = 0$, which is not possible if $P/Q$ is in its lowest terms. If $\lambda > \mu$, $x-\beta$ will be a factor if, and only if $B_1 = 0$, which also contradicts the assumption that $F/G$ is in its lowest terms. If $\lambda = \mu$, $x-\beta$ will be a factor if $B_1 = 0$, $C_1 = 0$, which is again contrary to assumption; but it is also a factor if $B_1 = -C_1$, and this is possible, even when $F/G$ and $P/Q$ are both in their lowest terms. In this case the rule given above would remove the terms in $B_1$ and $C_1$, and all succeeding pairs of terms such that $B_i + C_i = 0$, before the sum is reduced to a common denominator.

The proof for the general case, where the sum is of $n$ rational fractions, instead of two, and where the denominators contain $m_1, m_2, \cdots m_n$ binomial factors instead of two, is an obvious extension of the above.

When a fractional equation

$$\frac{F}{G}+\frac{P}{Q}=.0$$

is to be solved, it may not be apparent whether $F/G$ and $P/Q$ are in their lowest terms. Resolution into partial fractions will disclose the fact. If they are in their lowest terms, the usual method of solution, by reducing to lowest common denominator and equating to zero the sum of the numerators, will not yield an extraneous solution unless, in the above notation, $B_1 + C_1 = 0$; and the method suggested by the theorem just proved obviates this difficulty.

Example:

$$\frac{F}{G} = \frac{2x^2-4x+1}{(x-1)^2(x-2)} = \frac{1}{(x-1)^2}+\frac{1}{x-1}+\frac{1}{x-2},$$

$$\frac{P}{Q} = \frac{5x^2-15x+12}{(x-1)^2(x-3)} = \frac{-1}{(x-1)^2}+\frac{2}{x-1}+\frac{3}{x-3}.$$

Here both of the given fractions are in their lowest terms. The usual method of solving

$$\frac{F}{G}+\frac{P}{Q}=0$$

would be

$$\frac{F}{G}+\frac{P}{Q}=\frac{7x^3-35x^2+55x-27}{(x-1)^2(x-2)(x-3)}=0,$$

from which $7x^3-35x^2+55x-27=0$, or $x=1,\ 2\pm\frac{1}{7}\sqrt{7}$.

The root $x=1$ is extraneous. The sum of $F/G$ and $P/Q$ is not in its lowest terms. The reason is that $C_1=-B_1$. If the rule stated above is followed, one gets

$$\frac{F}{G}+\frac{P}{Q}=\frac{3}{x-1}+\frac{1}{x-2}+\frac{3}{x-3}=\frac{7x^2-28x+27}{(x-1)(x-2)(x-3)}=0,$$

which yields the true roots $x=2\pm\frac{1}{7}\sqrt{7}$.

## II. Determination of an Angle of a Right Triangle, without Tables.

By Roger A. Johnson, Hamline University.

If $a, b, c$ denote respectively the longer leg, the shorter leg, and the hypotenuse, then the value, in degrees, of the smaller acute angle is given approximately by

$$B=172\frac{b}{a+2c}.$$

This formula was given by Ozanam, 1699.

In practice it often happens that all three sides of a right triangle are known, and it is desired to find the angles quickly; the above formula will give them without the use of tables, with almost four-place accuracy for angles up to 35°, and better than three-place accuracy up to 45°. When the sides are given in integers, this method is simpler than the ordinary one. Again, it is useful as a check-formula in testing solutions.

The proof is easily effected by means of Taylor's Series. If we write

$$\frac{b}{a+2c}=\frac{\sin B}{2+\cos B}=f(B),$$

the expansion of $f(B)$ is easily found to be

$$f(B)=\frac{B}{3}\left(1-\frac{1}{180}B^4-\frac{1}{1512}B^6\cdots\right),$$

whence it is evident that for small values of $B$, $f(B)$ is nearly equal to $\frac{1}{3}B$ radians. That is, if we desire to have $B$ in degrees,

$$B\ (\text{degrees})=\frac{3\times 180}{\pi}f(B)+e,$$

where $e$ is a small correction. We may correct the error in part by replacing the coefficient $540/\pi$, or $171.89\cdots$, by the simpler number 172. The degree of

accuracy is best exhibited by working out the actual value given by the formula for various angles. The subjoined table shows that the angles given by the formula are too large, though the error scarcely exceeds .01 of a degree, up to about 33°; that thereafter the results are too small, and after the angle exceeds 45°, the discrepancy becomes rapidly larger. Since we can always use the formula to compute an angle less than 45°, this later divergence does not affect its usefulness.

| True Value. | Value by Formula. | True Value. | Value by Formula. |
|---|---|---|---|
| 5° | 5.0033 | 30° | 30.0067 |
| 10° | 10.0065 | 35° | 34.9945 |
| 15° | 15.0094 | 40° | 39.9703 |
| 20° | 20.0115 | 45° | 44.9270 |
| 25° | 25.0112 | 50° | 49.8562 |

## RECENT PUBLICATIONS.

### REVIEWS.

*General Theory of Polyconic Projections.* By OSCAR S. ADAMS, Geodetic Computer. Published by the Department of Commerce, U. S. Coast and Geodetic Survey, Serial No. 110, Special Publication No. 57, Washington, D. C., 1919. 174 pages. Price 25 cents.

To quote from the author's preface, "In this publication an attempt has been made to gather into one volume all of the investigations that apply to the system of polyconic projections." The author gives[1] Tissot's definition of a polyconic projection as "one in which the parallels of latitude are represented by arcs of a non-concentric system of circles with the centers of these various circles lying upon a straight line." Polyconic projections of the sphere and the ellipsoid of revolution only are considered, the whole purpose of the work being the construction of maps of the surface of the earth either as a whole or in part. The table of contents has thirty-one headings: Determination of ellipsoidal expressions, Development of general formulas for polyconic projections, Classification of polyconic projections, Rectangular polyconic projections, Stereographic meridian projection, Derivation of stereographic meridian projection by functions of a complex variable, Construction of stereographic meridian projection, Table for stereographic meridian projection, Stereographic horizon projection, Derivation of stereographic horizon projection by functions of a complex variab'e, Proof that circles project into circles in stereographic projections, Construction of stereographic horizon projection, Solution of problems in stereographic projections, Conformal polyconic projections, Determination of the conformal projection in which the meridians and parallels are represented by circular arcs, Special cases of the projection, General study of double circular projections,

[1] Page 10.

Conformal double circular projections, Cayley's principle, Discussion of the magnification on the conformal double circular projection, Equivalent, or equal area, polyconic projections, Conventional polyconic projections, non-rectangular double circular projections, Projection of Nicolosi or globular projection, Projection of P. Fournier, Ordinary or American polyconic projection, Tissot's indicatrix, Tables of elements of the ordinary, or American polyconic projection, Transverse polyconic projection, Projection for the international map on the scale of 1 : 1 000 000, Tables for the projection of the sheets of the international map of the world.

The book contains forty-eight figures and a folded frontispiece giving a transverse polyconic projection of the North Pacific Ocean.

The author states in the preface that in the preparation of the volume the following works were especially consulted: M. A. Tissot, *Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques*, Paris, 1881; A. Germain, *Traité des Projections des Cartes Géographiques*, Paris, 1866(?); N. Herz, *Lehrbuch der Landkartenprojectionen*, Leipzig, 1885; W. W. Hendrickson, U. S. N., *Notes on Stereographic Projection*.

The work may be described on the whole as a careful and detailed discussion of all such representations of the earth's surface on a plane as come under the definition of polyconic projection which have been found useful in practical map making. In general it may be said that the purpose of the discussion of each projection considered is first to determine and prove properties of the map as a whole, second to derive formulas for the coördinates of the point in the plane map representing a point of the earth's surface of given latitude and longitude, third to determine the scale or magnification of the map at any point in the directions of the circles of latitude and longitude. Both geometrical and analytical methods are extensively used. Not much mathematical knowledge is required of the reader, indeed we suppose the book to be intended rather for the practical maker of maps than for the mathematician. The author says, in the preface "it is hoped that the treatment may be found complete enough to serve all practical purposes." The figures are good, the book is well printed and remarkably free from misprints; we have noticed but two: on page 18, line 8, for $M$ read $N$, on page 156, line 10 from the bottom, for "about" read "by."

Two paragraphs we find of particular interest, the analytical discussion of rectangular polyconic projection, pages 13 to 18, and the determination by the use of the complex variable of the conformal projection in which both meridians and parallels are represented by circular arcs, pages 80 to 86, presumably after Lagrange.

Considered as a mathematical treatise the book has some faults. There is an almost total lack of exact references, except to the author's earlier publication, *General Theory of Lambert Conformal Projection*, Special Publication No. 53, U. S. Coast and Geodetic Survey. It would add greatly to the interest of the work to give the names of the originators of the various projections with dates and exact references. A more serious fault is the nearly complete absence of any

attempt to explain the special advantages of the different projections or the particular purposes for which they were devised. We find rarely even so little of this kind of information as is contained in the following statements: "This projection has been much used by the English War Office for the construction of maps," [1] or "The projection seems to have been devised by Supt. F. R. Hassler to meet the requirements in the charting of the coast of the United States." [2] It would be of interest to know what projections, proofs and discussions are original with the author.

We find the treatment uneven. Certain quite elementary problems, such as finding the integral, $\int \sec x \, dx$, are solved with wholly unnecessary detail, not once but several times, while other questions are handled sometimes without proper mathematical rigor and sometimes in an awkward way; some of the more technical map work lacks clearness. Thus the proof that in stereographic projection any circle of the sphere becomes a circle in the plane covers three pages[3] and is not very easy to follow. With the well-known formulas for the projection the proof may be given in a few lines. In the applications of the complex variable to the derivation of stereographic meridian projection and to stereographic horizon projection[4] the necessary functions, which are not very simple, are given and proved correct. It would be more natural to derive these functions by forming $x + iy$ from the known $x$ and $y$, a procedure which must, we think, be that actually followed. In the presentation of Lagrange's work on the determination of the conformal projection in which the meridians and parallels are represented by circular arcs an unknown constant is taken[5] as $\beta^2$, and $\beta$ is treated as real. There is no reason that the constant should be positive. In fact if $\beta^2$ is replaced by $-\beta^2$ the only effect is to interchange the $\lambda$ and $\sigma$ in the discussion. We have noticed one more serious omission: in the treatment of conformal polyconic projections[6] it is shown that the expression

$$u^2 \frac{u\frac{d\rho}{d\varphi} - \rho\frac{du}{d\varphi}}{u\frac{d\rho}{d\varphi} + \rho\frac{du}{d\varphi}}$$

is constant and that by a change of variable this expression is multiplied by an arbitrary constant. It is then stated that the expression may therefore be assumed equal to unity. The reasoning is correct unless

$$u\frac{d\rho}{d\varphi} - \rho\frac{du}{d\varphi} = 0.$$

[1] Page 18, referring to rectangular polyconic projection.
[2] Page 143, referring to American polyconic projection.
[3] Pages 43 to 46.
[4] Pages 30 and 42.
[5] Page 84.
[6] Pages 72 to 80.

By failing to consider this possibility one solution of the problem considered is lost, that is $s = \rho$, giving in the map circles of latitude passing through one point.

JAMES K. WHITTEMORE.

*Lezioni di Calcolo Infinitesimale dettata nella R. Università di Bologna e redatte per uso degli studenti.* S. PINCHERLE. Seconda edizione riveduta. Bologna, N. Zanichelli, 1920. 8vo. 8 + 785 pp. Price 40 lire.

Translation of an extract from the "avvertenza alla seconda edizione": "The lectures on the infinitesimal calculus which I gave to the press at the end of 1915, not without fear and trembling, have won favor, beyond all expectations, with the mathematical public, so that the call for a new edition came too soon to permit those modifications and additions that I had in mind to introduce.

"The second edition differs from the first only by slight changes, and only those parts have been retouched in which greater clarity or precision of statement seemed to me necessary."

Contents—Introduction, 1–50; Section first: Differential calculus, 51–302; Section second: Integral calculus, 303–484; Section third: Geometrical applications of infinitesimal calculus, 485–630; Section fourth: Differential equations, 631–764. There is a 14-page alphabetical index.

*Leçons de Géométrie Supérieure.* Par E. VESSIOT. Edition revue et augmentée. Avec une préface de M. G. Koenigs. Paris, J. Hermann, 1919. 10 + 376 pp. Price 30 francs.

Preface by E. Vessiot: "La première édition de ces leçons autographiée [1906], ayant été rapidement épuisée, j'ai accepté l'offre de réimpression que m'a faite M. Hermann. Les fautes d'impression avaient été corrigées par M. Anzemberger en vue de cette réédition. J'ai revu et amélioré la rédaction; et j'y ai fait des additions importantes. . . ."

Contents—I: Révision des points essentiels de la théorie des courbes gauches et des surfaces développables, 1–18; II: Surfaces, 19–34; III: Etude des éléments fondamentaux des courbes d'une surface, 35–60; IV: Les six invariants, La courbure totale, 61–81; V: Surfaces réglées, 82–120; VI: Congruences de droites, 121–160; VII: Congruences de normales, 161–189; VIII: Les congruences de droites et les correspondances entre deux surfaces, 190–237; IX: Les complexes de droites et les équations aux dérivées partielles du premier ordre, 238–268; X: Complexes linéaires, 269–292; XI: Transformations de contact, Transformations dualistiques, Transformations de Sophus Lie, changeant les droites en sphères, 293–315; XII: Systèmes triple orthogonaux, 316–325; XIII: Congruences de sphères et systèmes cycliques, 326–354; Exercices, 355–371.

*Physical Bases of Ballistic Table Computations. Ordnance Textbook.* [By Professor A. A. BENNETT.] (War Department, Document 972). Washington, Government Printing Office, 1920. 4to. 17 pp.

This monograph constitutes Part I of the Introduction of New Ballistic Tables being prepared by the Ordnance Department.

Quotation from 'Prefatory Remarks': "These new tables for exterior ballistics . . . were designed and supervised by the author of this introduction. The circumstances demanding their construction will not be reviewed in this part, nor will any account be given here of their form and content, nor of the technique and special devices used in their computation. It is only the physical facts and theories upon which these tables are founded that will here be treated, and even the history of these theories will be left with little more than mention.

"The matter treated here is qualitative rather than quantitative. The methods of numerical integration used in computing the trajectories, while in themselves only methods of approximation, are capable of giving results with any preassigned degree of accuracy, and as used yield vastly more precise figures than justified by the physical data available or the physical assumptions employed. This precision is obtained, however, at no extravagant sacrifice of labor, and secures results which are conveniently regular. Less accurate methods hitherto in vogue are now insufficient. The total number of physical factors in the problems of exterior ballistics is practically infinite. The elimination of all but a few is justified only by careful quantitative

experiments and computations which are not discussed here. To catalogue and describe, merely, the various influences affecting the projectile in flight, leaves the relative importance assigned to each apparently hap-hazard and arbitrary. Thus this introduction seeks to mention only those physical assumptions which enter explicitly into the computations of the accompanying tables, and attempts only such justification for them as is in keeping with the expository nature of this paper."

*Solid Geometry with Problems and Applications.* Revised edition. By H. E. SLAUGHT and N. J. LENNES. Boston, 1919. 12mo. 8 + 211 pp. Price $1.00.

Quotations from the Preface: "In re-writing the *Solid Geometry* the authors have consistently carried out the distinctive features described in the preface of the *Plane Geometry*. . . . Owing to the greater maturity of the pupils it has been possible to make the logical structure of the *Solid Geometry* more prominent than in the *Plane Geometry*. The axioms are stated and applied at the precise points where they are to be used. Theorems are no longer quoted in the proofs but are only referred to by paragraph numbers; while with increasing frequency the student is left to his own devices in supplying the reasons and even in filling in the logical steps of the argument. For convenience of reference the axioms and theorems of plane geometry which are used in the Solid Geometry are collected in the Introduction.

"In order to put the essential principles of solid geometry, together with a reasonable number of applications, within limited bounds (156 pages), certain topics have been placed in an Appendix. This was done in order to provide a minimum course in convenient form for class use and not because these topics, Similarity of Solids and Applications of Projection, are regarded as of minor importance. In fact, some of the examples under these topics are among the most interesting and concrete in the text. . . .

"The treatment of incommensurables throughout the body of this text, both Plane and Solid, is believed to be sane and sensible. In each case, a frank assumption is made as to the existence of the concept in question (length of a curve, area of a surface, volume of a solid) and of its realization for all practical purposes by the approximation process. Then, for theoretical completeness, rigorous proofs of these theorems are given in Appendix III, where the theory of limits is presented in far simpler terminology than is found in current text books and in such a way as to leave nothing to be unlearned or compromised in later mathematical work."

*Plane Trigonometry for Secondary Schools.* By C. DAVISON. Cambridge: at the University Press, 1919. 12mo. 4 + 334 pp. Price 6 shillings and 6 pence.

Contents—I: Trigonometrical ratios of an acute angle, 1–30; II: Circular measure, 31–40; III: Circular functions of an angle of any magnitude, 41–56; IV: Graphs of the circular functions, 57–65; V: Circular functions of compound angles, 66–84; VI: Circular functions of multiple angles, 85–98; VII: Transformation of trigonometrical expressions, 99–111; VIII: Solution of trigonometrical equations, 112–141; IX: Inverse functions, 142–147; X: Circular functions of sub-multiple angles, 148–158; XI: The geometry of the triangle, 159–174; XII: The solution of triangles, 175–187; XIII: Practical applications, 188–208; XIV: The principal circles associated with a triangle, 209–223; XV: The geometry of the quadrilateral, 224–232; XVI: Areas of regular polygons and circles, 233–238; XVII: Inequalities, 239–247; XVIII: Approximations and errors, 248–256; XIX: DeMoivre's theorem, 257–274; XX: Series, 275–287; Problem papers, 288–308; Answers, 309–334.

*Elementary Applied Mathematics. A practical course for general students.* By W. P. WEBBER. New York, Wiley, 1920. 12mo. 10 + 115 pp. Price $1.25.

Preface: "This course is a response to a demand in this university [of Pittsburg]. It is an effort to provide a course that is complete in itself and sufficiently general and practical to meet the needs of a large class of students who are not to specialize in mathematics but who do want some elementary mathematical training that they can use in everyday affairs.

"While the course has been developed simultaneously with Webber and Plant's *Introductory Mathematical Analysis* and there is some material common to the two courses, yet they are addressed to distinctly different groups of students.

"A knowledge of elementary algebra and geometry is presupposed. Students with little or no knowledge of formal geometry have succeeded with the course.

"The course has been given by lecture or in mimeograph form to classes during the past several years. . . ."

Contents—I: Review of elementary algebra, 1–9; II: Geometric theorems, 10–13; III: Methods of calculation, 14–27; IV: Graphic representation, 28–41; V: Ratio, proportion and variation, 42–47; VI: Geometric problems, 48–54; VII: Rectangular coördinates, graphs of equations, empirical formulas, 55–71; VIII: Applications of percentages, 76–79; IX: Analysis of food and receipts, 80–85; X: Individual and family accounts, 86–96; Tables 97–115.

## ARTICLES IN CURRENT PERIODICALS.

**AMERICAN MACHINIST,** New York, volume 52, March 25, 1920: "War on the decimal system" by F. Franz, 682–684; "Octaval notation and the measurement of binary inch fractions" by A. Watkins, 685–688.

**ANNALS OF MATHEMATICS,** second series, volume 21, no. 3, March, 1920: "A Green's theorem in terms of Lebesgue integrals" by H. E. Bray, 141–156; "Bilinear operations generating all operations rational in a domain $\Omega$" by N. Wiener, 157–165; "On the enumeration of proper and improper representations in homogeneous forms" by E. T. Bell, 166–179; "A proof of Jordan's theorem about a simple closed curve" by J. W. Alexander, 180–184; "Linear order in three dimensional Euclidean and double elliptic spaces" by G. H. Hallett, Jr., 185–202; "Further properties of the general integral" by P. J. Daniell, 203–220; "Summability of double series" by L. L. Smail, 221–223; "The fundamental theorem of celestial mechanics" by J. L. Coolidge, 224.

**BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY,** volume 26, no. 7, April, 1920: "Parametric equations of the path of a projectile when the air resistance varies as the $n$th power of the velocity" by F. H. Safford, 289–293; "Infinite systems of functions" by W. E. Milne, 294–300; "On certain related functional equations" by W. H. Wilson, 300–312; "The equation $ds^2 = dx^2 + dy^2 + dz^2$" by E. T. Bell, 312–318; "A property of permutation groups analogous to multiple transitivity" by W. B. Carver and Mrs. Estella F. King, 319–322; "College algebras" by Elizabeth B. Cowley, 323–329 [review of E. B. Skinner's *College Algebra* (New York, Macmillan, 1918), of W. C. Brenke's *Advanced Algebra* (New York, Century, 1917) and of Helen A. Merrill and Clara E. Smith's *A First Course in Higher Algebra* (New York, Macmillan, 1917)]; "Notes," 329–333; "New publications," 334–336.

**CANADIAN CHARTERED ACCOUNTANT,** Toronto, Ontario, volume 9, no. 4, April, 1920: "Bond schedules for the amortization of a premium or accumulation of a discount" by S. D. Killam, 231–236.

**EDUCATIONAL REVIEW,** volume 59, no. 4, April, 1920: "Junior high school mathematics" by T. Lindquist, 296–303. [Quotation: "Students who have completed a satisfactory junior high school course in mathematics should be able during their senior high school to complete plane and solid geometry in one year, physics in one year, and algebra III together with trigonometry or college algebra the final year."]

**ENGINEERING AND CONTRACTING,** Chicago, volume 53, February 4, 1920: "Instruction and tables for reducing labor in curve computation" by J. H. Lilly, 137–138.

**ENGINEERING NEWS-RECORD,** New York, volume 84, February 19, 1920: "Solution of compound curve problems" by L. R. Brown, 378–379 [Reprinted from *Electric Railway Journal*, Jan. 17, 1920]—March 4: Table for converting from scale 1 in. = 100 ft. in plotting curves," 486.

**INDUSTRIAL MANAGEMENT,** volume 59, no. 4, April, 1920: "The mathematics of labor turnover" by C. G. Barth, 315–318.

**INTERNATIONAL MARINE ENGINEERING,** New York, volume 25, March, 1920: "Pointing off when using the slide rule" by B. W. Manier, 228–229.

**JOURNAL OF EDUCATIONAL RESEARCH,** University of Illinois, volume 1, no. 3, March, 1920: "A shorter method for computing the coefficient of correlation" by L. P. Ayres, 216–221; "A bibliography of standard tests for the high school" by W. S. Monroe, 240–242 [VIII. Mathematics].

**MACHINERY,** New York, volume 26, January, 1920: "Development of time slide-rule" by J. B. Conway, 453–454; "To draw a circle tangent to three given circles" by C. N. Pickworth, 459–460—March: "Computation of diameters and circumferences of circles" by E. T. Meehan, 656.

**MATHEMATICS TEACHER,** volume 12, no. 3, March, 1920: "Association of Mathematics Teachers of New Jersey: Report of the Committee of First-Year High-School Mathematics," 89–100; "Proposed syllabus in algebra" by C. F. Wheelock, 101–114; "Mathematics for the physiologist and physician" by H. B. Williams, 115–123 [Paper read before the Mathematical Association of America, January 1, 1920]; "Love mathematical" by R. C. Gillies, 124–125; "New Books," 126–127.

**MIND,** new series, no. 114, April, 1920: Review by C. D. Broad of A. N. Whitehead's *The Principles of Natural Knowledge* (Cambridge, 1919), 216–231.

**MONIST,** volume 30, no. 2, April, 1920: "Philip Bertrand Jourdain," 161–182 + frontispiece portrait [Nearly half of this appreciation is from "memories" written by Jourdain's youngest sister Milly; the article concludes with a list of thirty-one articles written for the *Monist*, 1908–1920]; "Elliptic orbits and the growth of the third law of Newton" by P. E. B. Jourdain, 183–198; "Newton's theorems on the attraction of spheres" by P. E. B. Jourdain, 199–202; "Leonardo da Vinci (Born 1452. Died 1519)" by M. Jourdain, 280–291.

**PHILOSOPHICAL MAGAZINE,** sixth series, volume 39, April, 1920: "Applications of quaternions to the theory of relativity" by H. T. Flint, 439–449—May: "On the advance of the perihelion of a planet, and the path of a ray of light in the gravitation field of the sun" by A. Anderson, 626–628; Review of A. N. Whitehead's *An Enquiry concerning the Principles of Natural Knowledge* (Cambridge, 1919), 629–631 [Last paragraphs: "It will be seen then that this book is simply invaluable to anyone who wishes to bring himself into line with the new principle of relativity, whether his interest be scientific in the narrow sense or philosophical in the wide sense.

"John Stuart Mill tells us in his Autobiography that he was at times actually depressed by the thought that musical chords though practically infinite in the number of combinations they admitted were yet in reality finite and exhaustible. Our feeling as we close Professor Whitehead's book is one almost of elation at the thought of how little we know, and how uncertain is the little we think we know, when we form our concepts of the framework of infinite Nature."]

**REVUE DE L'ENSEIGNEMENT DES SCIENCES,** volume 13, nos. 127–130, July–December, 1919: "Enveloppes des courbes et des surfaces à un paramètre" by F. Meyer, 177–186; "Sur les fonctions linéairement distinctes" by P. Montel, 186–190; "Sur la surface du triangle" by P. Montel, 190–197; "Sur l'équation en nombres entiers $a^2 + b^2 = c^2$" by A. Lévy, 197–211; "Limite de $(1 + 1/x)^x$ pour $x$ infini" by R. Dontot, 211–214; "Questions de forme" by J. Juhel-Rénoy, 215–218; "Problèmes de mathématiques donnés au baccalauréat en Octobre 1918," 218–223, 229–234; "Examens et Concours de 1919: Agrégation des sciences mathématiques et Ecole Normale Supérieure," 241–251.

**REVUE DE MATHÉMATIQUES SPÉCIALES,** volume 13, no. 1, October, 1919: "Sur la somme des puissances semblables des $n$ premiers nombres entiers" by R. Dontot, 1–4; Solutions of questions in algebra, analytic geometry, analysis, and mechanics, 4–10, 17–22; Questions in the written and oral examinations for the Ecole Polytechnique, 1919, 10–15; Ecole Centrale Concours de 1919, 23–24—No. 2, November: "Sur la somme des puissances semblables des $n$ premiers nombres entiers" (suite) by R. Dontot, 25–28; Problems proposed and solved in examinations for the Ecole des Ponts et Chaussées, Ecole Polytechnique, Ecole Centrale, etc., 29–56—No. 3, December: "Sections planes et sphériques du tore" by A. Sainte-Laguë, 57–67; Problems and Solutions, 60–80—No. 4, January, 1920: "Sections planes et sphériques du tore" (fin) by A. Sainte-Laguë, 81–83; Problems and Solutions, 83–104—February, 1920: "Enveloppe des droites obtenus en faisant tourner d'un angle constant, $\alpha$, les tangentes à une courbe plane autour de leurs points de contact" by Jean Mourret, 105–106; Problems and solutions, 107–128.

**REVUE DU MOIS,** volume 15, January, 1920: "Radioactivité, probabilité et déterminisme" by E. Borel, 33–40— March: "La réforme de l'enseignement et les anciens combattants" by E. Borel, 225—227.

**REVUE SCIENTIFIQUE,** April 10, 1920: "La vie et l'oeuvre de Lord Kelvin" by E. Picard, 193–207.

**SCHOOL AND SOCIETY,** volume 11, March 20, 1920: "The National Committee on Mathematical Requirements," 343–344—April 17, 1920: "The Bode theory of transfer applied to the teaching of mathematics" by E. B. Lytle, 457–462.

**SCIENCE PROGRESS,** volume 14, April, 1920: Recent advances in pure mathematics, by Dorothy M. Wrinch, 536–543; Recent advances in applied mathematics, by S. Brodetsky, 543–550; "Octaval notation for inch fractions," 646–647 [Quotation: "A rule for engineer's precision measurements has just been issued which embodies a proposal for a logical notation for the inch binary fractions. Such notation is not new, but for the first time is put into a practical form for the workshop. . . . In place of the radix of 10 used for decimal notation, this rule . . . uses a radix of 8. Just as in decimals $0.234 = \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3}$ . . . so in octavals $0_8234 = \frac{2}{8} + \frac{3}{8^2} + \frac{4}{8^3}$;"] "Logic and mathematics" by the late P. E. B. Jourdain [review of J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago and London, 1918) and of B. Russell's *An Introduction to Mathematical Philosophy* (London and New York), 669–674; Reviews by Dorothy Wrinch of H. Lamb's *An Elementary Course of Infinitesimal Calculus* (3d edition, Cambridge, 1919), and of L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun's *Unified Mathematics* (Boston and London, 1919), 678–679.

**SCIENTIA,** volume 27, no. 3, March, 1920: Review by G. Loria of H. Bateman's *Differential Equations* (London, 1918), 228–229; Review by W. J. Greenstreet of L. Huxley's *Life and Letters of Sir Joseph Dalton Hooker* (London, 1918), 236–239.—No. 4, April: Review by G. Loria of E. Bortolotti's *Italiani scopritori e promotori di teorie algebriche* (Modena, 1919), 315–317; Review by R. Morcolongo of P. Appell and S. Dautheville's *Précis de mécanique rationnelle* (2e éd., Paris, 1918), 317–318.

**TECHNOLOGY REVIEW,** Cambridge, Mass., volume 22, no. 2, April, 1920: "Richard Cockburn Maclaurin" by J. P. Munroe, 252–262; "Resolutions by the Alumni Council," 263–264 [regarding R. C. Maclaurin]; "The work of Professor Edward C. Pickering at the Massachusetts Institute of Technology, 1867–1877" by C. R. Cross, 277–288 [Professor Pickering was Thayer professor of physics at the Institute for the ten years preceding his appointment as professor of astronomy and director of the Harvard Observatory in 1876; he established the first physical laboratory in the United States]; "Joseph John Skinner, Died November 12, 1919" by W. T. Sedgwick, 289–290 + portrait [Professor Skinner, who was assistant professor in mathematics at M. I. T. 1896–1904, was born in Vermont in January, 1842. "From 1874 to 1881 he taught mathematics at the Sheffield Scientific School, and also some classes in physics and astronomy. One of his pupils in 1874 was the writer, whose recollections of 'Tutor Skinner' are of a man always genial and especially courteous to his students. It was in his classes that some students were for the first time in their lives treated as gentlemen and equals, and the reaction was encouraging and stimulating. On the other hand 'Tutor Skinner' did not escape chaffing in a familiar rhyme which ran:

> "There was a bold tutor and he was a sinner.
> His first name was Joseph, his last name was Skinner.
> He wrestled the Freshmen and brought them to time,
> With his tangent, co-tangent, co-secant, co-sine."]

**TÔHOKU MATHEMATICAL JOURNAL,** volume 17, nos. 1–2, February, 1920: "On the conservative field of force" by K. Ogura, 1–6; "A substitute for Duhamel's theorem" by G. James, 7–9; "Ueber Kriterien für Irreduzibilität ganzzahliger algebraischer Gleichungen" by M. Fujiwara, 10–17; "On the law of errors in the space of $p$ dimensions" by A. Guldberg, 18–23; "Aeronautical photogrammetry with an appendix on reconnoitring photogrammetry" [in Japanese] by T. Iwat-uki, 24–63; "On a certain transcendental integral function in the theory of interpolation" by K. Ogura, 64–72; "Sur les polygones podaires d'un polygone plan donné" by V. Thébault, 73–83; "On the converse of the theorem of Scheeffer's" by Y. Uchida, 84; "Proof of the theorems due to Messrs. Kojima and Okada" [in Japanese] by M. Watanabe, 85–87 "Groups of order $g$ containing $(g/2) - 1$ involutions" by G. A. Miller, 88–102; "On some methods of construction in elementary geometry of three dimensions" by K. Yanagihara, 103–108; "On the nature of the integrals of a certain linear differential equation" by K. Ôishi, 109–117; "Sur des points rémarquables du triangle" by V. Thébault, 118–122; "Remarks on the note 'On the Fourier constants'" by M. Plancherel and K. Ogura, 123–128; "On the theory of interpolation" by K. Ogura, 129–145.

**UNIVERSITY BULLETIN, LOUISIANA STATE UNIVERSITY,** new series, volume 12, no. 3: March, 1920: "The mathematical process and some of its disciplines" by S. T. Sanders, 31 pp.

## AMERICAN DOCTORAL DISSERTATIONS.

H. J. ETTLINGER, "Existence theorems for the general real self-adjoint linear system of the second order," *Transactions of the American Mathematical Society*, vol. 19, January, 1918, pages 79–86. (First part of dissertation, Harvard, 1919).

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# PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

**Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.**

## PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them*. If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

**2850. Proposed by SARAH BEALL, U. S. Coast and Geodetic Survey.**

An unknown star is observed at the altitudes $h_1$ and $h_2$ at the respective times $t_1$ and $t_2$, the latitude being known also. Obtain formulas for the right ascension and declination of the star: (1) when the time interval $t_2 - t_1$ is large: (2) when the time interval is so small that $(h_2 - h_1)/(t_2 - t_1)$ may be taken as the value of $dh/dt$ corresponding to the mean altitude $(h_1 + h_2)/2$ and the mean time $(t_1 + t_2)/2$. This problem sometimes arises when a bright star is observed through the clouds.

**2851. Proposed by HILLEL PORITSKY, Cornell University.**

Does there exist an analytic function, satisfying the functional equation, $f(z + 1) = e^{f(z)}$?

**2852. Proposed by D. H. RICHERT, Bethel College, Newton, Kan.**

What is the radius of a cylinder inscribed in a right cone, radius of base $R = 5$ inches, and altitude $h = 18$ inches, the volume of the cylinder to be $(1/n = 3/4)$ that of the cone?

**2853. Proposed by J. S. BROWN, Southwest Texas State Normal College, San Marcos, Texas.**

Find the side and apothem of a regular pentagon inscribed in a circle, without the use of extreme and mean ratio.

**2854. Proposed by C. N. MILLS, Heidelberg University.**

Solve the simultaneous equations for $x$ and $y$,

$$x^n + y^n = a_n, \qquad x^{n-1} + y^{n-1} = a_{n-1}.$$

**2855. Proposed by J. L. RILEY, Stephenville, Texas.**

Show that the circle of curvature at any point of the ellipse cannot pass through the centre unless the eccentricity be greater than $1/\sqrt{2}$.

**2856. Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey.**

Show that for the real domain defined by $+1 > x > -1$, $s$ a positive integer,

$$\frac{1}{(1-x^s)^{1/s}}\int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{n=\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)}\, x^{ns+1}$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}}\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{n=\infty} \frac{n!\, s^n}{(s+1)(2s+1)\cdots(ns+1)}\, x^{ns+1}.$$

**2857. Proposed by the late L. G. WELD.**

A savings bank offers to pay 3% interest on deposits, said interest to be continuously compounded, *i.e.*, compounded at infinitesimal intervals of time. What would be the amount of $1.00 for one year?

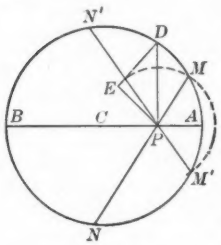
## SOLUTIONS OF PROBLEMS.

**2753 [1919, 73]. Proposed by A. M. HARDING, University of Arkansas.**

Through a point $P$, within a circle, draw a chord which will be trisected at $P$.

### I. Solution by P. J. da Cunha, University of Lisbon.

Soient $C$ le centre et $R$ le rayon du cercle donné, $AB$ le diamètre mené par le point intérieur $P$, $a$ la distance $CP$ et $MN$ la corde demandée. Comme sur toutes les cordes menées par un point intérieur à un cercle donné, les deux segments compris entre le point et la circonférence ont un produit constant, on a $MP \times PN = AP \times PB$, ou, $MP$ devant être la tiers de $MN$,




$$2MP^2 = (R - a)(R + a) = R^2 - a^2.$$

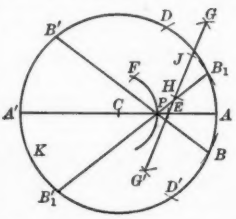
Alors, si l'on mène par $P$ une perpendiculaire au diamètre $AB$, cette perpendiculaire coupe la circonférence à un point $D$, et l'on a $DP^2 = R^2 - a^2$, et, par conséquent, $2MP^2 = DP^2$.

Pour déterminer $MP$ il suffit, donc, de construire un carré dont la diagonale soit $DP$. Soit $PE$ le coté de ce carré. Du point $P$ comme centre, avec un rayon égale à $PE$, on décrit un arc qui coupe la circonférence en deux points $M$ et $M'$. Les cordes $MN$ et $M'N'$ repondent à la question.

### II. Solution and Discussion by E. J. Moulton, Northwestern University.

It is obviously not always possible to construct such a chord; for example, there is no such chord if $P$ is at the center of the circle.

Let $K$ be the given circle, $C$ its center and $AA'$ the diameter through $P$. If there exists a chord having the desired property let $B'B$ be such a chord, the notation chosen so that $B'P = 2BP$. Then we shall have $B'P \cdot BP = A'P \cdot AP$ or $2BP^2 = (AC + CP)(AC - CP) = AC^2 - CP^2$. Now we may construct a segment $x$ such that $2x^2 = AC^2 - CP^2$ as follows: Draw a right triangle with hypotenuse equal to $AC$ and one leg equal to $CP$; then the length, $y$, of the other leg is such that $y^2 = AC^2 - CP^2$. Next construct an isosceles right triangle with hypotenuse equal to $y$; then the legs will be of the required length $x$, since we have $2x^2 = y^2$. Hence, if the required chord exists its shorter segment is of the constructed length $x$. If $x < PA$, assuming $PA < PA'$, it therefore follows that no such chord exists. If $x \geqq PA$, the extremity $B$ of the chord may be located by drawing a circle with $P$ as center and $x$ as radius, and noting the points of intersection of this circle with the given circle $K$; any such intersection is a possible position for $B$.




The complete ruler and compass construction apparently is made most briefly as follows: Having given the circle $K$ with center at $C$ and the point $P$, draw the diameter $l_1 = A'A$ through $C$ and $P$, with $P$ nearer $A$ than $A'$. With $A$ as center and $AC$ as radius draw a circle $K_1$ intersecting[1] $K$ in the distinct points $D$ and $D'$. Draw the line $l_2 = DD'$ intersecting[1] the line $l_1$ in the point $E$. With $E$ as center and radius $EA$ draw the circle $K_2$; with $C$ as center and radius $CP$ draw the circle $K_3$. Let $F$ be one of the points of intersection[1] of the circles $K_2$ and $K_3$. Draw the line $l_3 = AF$. With $AF$ as radius,[2] draw the circle $K_4$ with center at $A$, and the circle

[1] The statement here is put in the usual form of elementary geometry. It is to be noticed that there is a serious defect in the logic here if we fail, as Euclid does under similar circumstances, to prove that the given circles or lines actually do intersect in the required number of points. It can be proved, although the necessary axioms and theorems are not at hand in Euclid, that in the construction we are making the required points of intersection exist in all cases where there is a reference to this foot-note, the proofs being based on axioms and theorems given, for example, by Veblen in Young's *Monographs on Modern Mathematics*.

[2] Here we are simply bisecting the segment $AF$. There is a logical objection to a common form of procedure, where arcs of unspecified radius are drawn, because it cannot be proved that

$K_5$ with center at $F$. Let $G$ be one of the points of intersection[1] of $K_4$ and $K_5$. Draw the line $l_4 = GE$ intersecting[1] the line $l_3$ at the point $H$. With $H$ as center and radius $HA$ draw the circle $K_6$; let $J$ be one of its intersections with the line $l_4$. With $AJ$ as radius and $P$ as center draw the circle $K_7$ intersecting[3] the circle $K$ in the points $B$ and $B_1$. Draw the chords $l_5 = BPB'$ and $l_6 = B_1PB_1'$. These are the required chords. Our construction has required the drawing of six lines $l_1$, $l_2$, $l_3$, $l_4$, $l_5$, $l_6$ and seven circles $K_1$, $K_2$, $K_3$, $K_4$, $K_5$, $K_6$, $K_7$, the locating of ten necessary points $A$, $B$, $B_1$, $D'$, $D$, $E$, $F$, $G$, $H$, $J$. (If one assumes that the center $C$ is not given, as one might from the statement of the problem, the construction is much more difficult.)

Also solved by GEORGE AGINS, E. H. CLARKE, H. N. CARLETON, H. H. DOWNING, EMANUEL GOLDFARB, E. D. GRANT, LAURA GUGGENBUHL, R. A. JOHNSON, MARCIA L. LATHAM, E. W. MARTIN, F. V. MORLEY, A. G. MONTGOMERY, H. L. OLSON, W. B. PIERCE, ARTHUR PELLETIER, JOSEPH ROSENBAUM, ELIJAH SWIFT, CHARLES SCHUMAN, L. G. WELD, C. C. YEN, and the PROPOSER.

**2754 [1919, 73]. Proposed by J. W. LASLEY, JR., University of North Carolina.**

Given $\bar{x} = \tan^{-1}\dfrac{z}{\sqrt{x^2+y^2}}$, $\bar{y} = \tan^{-1}\dfrac{y}{x}$, $\bar{z} = \log\sqrt{x^2+y^2+z^2}$, solve for $x$, $y$, and $z$ in terms of $\bar{x}$, $\bar{y}$, and $\bar{z}$.

SOLUTION BY E. S. SMITH, University of Cincinnati

From the given equations, we have $z/(\sqrt{x^2+y^2}) = \tan\bar{x}$ (1), $y/x = \tan\bar{y}$ (2), and

$$x^2 + y^2 + z^2 = e^{2\bar{z}}. \tag{3}$$

Substituting the value of $y$ from (2) in (1), gives

$$z = x\sec\bar{y}\tan\bar{x}. \tag{4}$$

Substituting the values of $y$ and $z$ from (2) and (4) in (3), we have

$$x = \pm e^{\bar{z}}\cos y\cos\bar{x}. \tag{5}$$

Hence, from (2) and (5),

$$y = \pm e^{\bar{z}}\sin\bar{y}\cos\bar{x}. \tag{6}$$

From (4) and (6), we have

$$z = \pm e^{\bar{z}}\sin\bar{x}. \tag{7}$$

Hence, the result is $x = \pm e^{\bar{z}}\cos\bar{x}\cos\bar{y}$, $y = \pm e^{\bar{z}}\sin y\cos\bar{x}$, and $z = \pm e^{\bar{z}}\sin\bar{x}$.

Also solved by MARCIA L. LATHAM, E. W. MARTIN, H. L. OLSON, GEORGE PAASWELL, ARTHUR PELLETIER, C. H. RICHARDSON, and D. L. STAMY.

**2755 [1919, 73]. Proposed by J. L. RILEY, Stephenville, Texas.**

Every number whose square is the sum of the squares of two consecutive integers is equal to the sum of the squares of three integers of which two, at least, are consecutive.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

It is well known that the solution of the equation $a^2 + b^2 = c^2$, where $a$, $b$, $c$ are relatively prime integers, is given by the formulas, $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, where $m$ and $n$

points corresponding to $G$ exist. The radius most easily specified for which the existence of $G$ can be proved is the radius $AF$. It would not do, for example, to say "let us take any radius greater than a half of $AF$" because that does not specify an exact radius and it would need to be proved that any radius which was used was actually "greater than a half of $AF$". The simplest radius to use for the construction and a logical proof is the radius $AF$.

[3] Here we have assumed as in previous cases that there are points of intersection, but in this case there are none unless $AP \leqq AJ$. Since $AJ$ is one-third of the required chord $BB'$, when it exists, it follows that $AP \leqq \frac{1}{3}BB'$, which is impossible if $AP$ is greater than one-third of a diameter of $K$. If $AP \leqq \frac{1}{3}$ of a diameter, there always exists a chord which is trisected at $P$.

are relatively prime. In the case above $a - b = \pm 1$, whence either (1) $(m-n)^2 - 2n^2 = 1$, or (2) $(m+n)^2 - 2m^2 = 1$. The equation (2) may be derived from (1) by interchanging $m$ and $n$ and changing the sign of $n$. This would not affect the value of $c$, nor the algebraic work below,[1] hence we need consider only (1).

Writing (1) in the form $(m-n)^2 - 1 = 2n^2$, and factoring we have

$$[m-n-1][m-n+1] = 2n^2. \tag{3}$$

Since the two factors differ by 2 and their product is even, each is even, 2 is the H.C.F., and one factor is twice an odd square and the other the square of an even number. Accordingly we have

$$(4)\quad \begin{matrix} m-n-1 = 2\alpha^2 \\ m-n+1 = 4\beta^2 \end{matrix} \qquad \text{or} \qquad (5)\quad \begin{matrix} m-n-1 = 4\beta^2 \\ m-n+1 = 2\alpha^2, \end{matrix}$$

where $\alpha$ and $\beta$ denote integers. From (3), (4) and (5) we find

$$n = 2\alpha\beta, \qquad m = \alpha^2 + 2\beta^2 + 2\alpha\beta, \qquad \pm 1 = 2\beta^2 - \alpha^2, \tag{6}$$

the upper sign resulting from equations (4): the lower, from (5). Substituting these values in the expression for $c$, we find

$$c = m^2 + n^2 = (\alpha^2 + 2\beta^2 + 2\alpha\beta)^2 + (2\alpha\beta)^2 = (\alpha^2 + 2\alpha\beta)^2 + (2\alpha\beta + 2\beta^2)^2 + (2\alpha\beta)^2.$$

But $(\alpha^2 + 2\alpha\beta) - (2\alpha\beta + 2\beta^2) = \mp 1$ from (6), so that $c$ is expressed in the desired form.

It is worthy of note that every solution of the equation $\lambda^2 - 2\mu^2 = \pm 1$ leads to a triangle having the given property, and such triangles can be obtained only from such solutions. All these can be found by developing $\sqrt{2}$ into a continued fraction and taking the numerator and denominator of any convergent as $\lambda$ and $\mu$ respectively. In this way we get the following set of solutions:

| $\lambda$ | $\mu$ | $m$ | $n$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 3 | 4 | $5 = 0^2 + 1^2 + 2^2$ |
| | | 5 | 2 | 21 | 20 | $29 = 2^2 + 3^2 + 4^2$ |
| 3 | 2 | 12 | 5 | 119 | 120 | $169 = 3^2 + 4^2 + 12^2$ |
| | | 29 | 12 | 697 | 696 | $985 = 12^2 + 20^2 + 21^2$ |
| 7 | 5 | 70 | 29 | 4059 | 4060 | $5741 = 20^2 + 21^2 + 70^2$. |

**2756 [1919, 73]. Proposed by PAUL CAPRON, U. S. Naval Academy.**

Given a parallelogram with center $O$, vertices $PQP'Q'$, mid-points of sides $ABA'B'$ (cyclic order $PAQBP'A'Q'B'$). Let $K$ be any point of $OA$. Draw $KLH$ parallel to $Q'Q$ cutting $AQ$ at $L$, and draw $BLM$, meeting $OA$ produced at $M$. Draw $MH$, parallel to $PP'$, to meet $KLH$ at $H$, and draw $B'KE$ to meet $BL$ at $E$. Repeat, changing $A, B, P, Q$ to $A', B', P', Q'$, respectively, and *vice versa*. Repeat each of the foregoing, changing $P, P', B, B'$ to $Q, Q', B', B$, respectively, and *vice versa*.

What are the loci of $E$ and $H$? Show that $EH$ passes through $A'$ when $K$ is a point of $OA$, through $A$ when $K$ is a point of $OA'$. Consider the effect of interchanging the rôles of $A$ and $B$.

(This construction, as commonly given, is specialized in these particulars: the parallelogram is rectangular, the divisions of $OA$ are equal, and the locus of $H$ is not found.)

## I. Solution by Arthur Pelletier, Montreal, Can.

Let $OB = a$ and $OA' = b$ be the axes of coördinates and, taking $OK = -mb$, we find the following equations of the indicated straight lines:

(1) $(KLH)$ $\quad x/a + y/b = -m,$ $\qquad$ (2) $(BLM)$ $\quad x/a - my/b = 1,$

(3) $(MH)$ $\quad x/a - y/b = 1/m,$ $\qquad$ (4) $(B'KE)$ $\quad x/a + y/mb = -1.$

By eliminating $m$ from (2) and (4) we find for the equation of the locus of $E$, $x^2/a^2 + y^2/b^2 = 1$. Hence when $K$ passes from $A$ to $O$ the point $E$ describes the part $AB$ of an ellipse inscribed in the parallelogram, tangent to the sides at $A, B, A', B'$. The other parts of the ellipse will evidently be obtained by making the changes indicated in the problem. Similarly, we obtain from (1) and (3) the equation $x^2/a^2 - y^2/b^2 = -1$. Hence the locus of $H$ is the part $AHC$ of an hyperbola tangent to $PQ$ at $A$ etc.

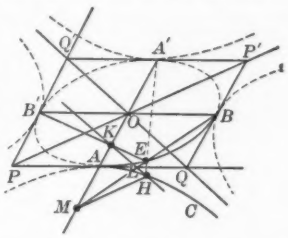
[1] The new $n$ would be negative, consequently $\alpha\beta$ in (6) would be negative. Two of the three numbers, the sum of whose squares is $c$, would be $2\alpha\beta - \alpha^2$, $2\alpha\beta - 2\beta^2$ and the difference is again $\pm 1$.

If we add to the two sides of (1) the corresponding sides of (3) multiplied by $m$, we obtain $(1+m)(x/a)+(1-m)y/b = 1-m$, the equation of a straight line through $H$. The same process applied to (2) and (4) yields the *same* equation and, since the point $(0, b)$ satisfies the new equation, we see that $H$, $E$, and $A'$ lie on this straight line.

*Note.*—A geometrical solution may be given. Starting with a square we easily find as loci, a circle and two conjugate equilateral hyperbolas. Then, by projection, we derive the loci found above. By the theorem of Menelaüs we prove that the correlative points $E$, $H$, and $A'$ are collinear, a property preserved in projection.

## II. Solution by Otto Dunkel, Washington University.

The ranges of points $K$ and $L$ are projective and hence $E$ is the intersection of two projective pencils with centers at $B'$ and $B$, such that $B'B$ is not self-corresponding. It thus follows that the locus of $E$ is a conic tangent at $B'$ and $B$ and passing through $A$ and $A'$ of the sides of the parallelogram. By using $A$ and $A'$ as centers it will be seen that it is tangent to the other two sides $PQ$ and $P'Q'$. Similarly, the range of points $M$ is projective with the ranges $L$ and $K$, and hence $H$ is the intersection of two pencils with centers at the points at infinity on $OQ$ and $OP$. Thus the locus of $H$ is a conic with $OQ$ and $OP$ as asymptotes, and the conic passes through $A$ and $A'$. It may be shown by the theory of conics (analytic or projective theory) that $PQ$ is a tangent. It also follows that the pencils $A'$ $(E)$ and $A'$ $(H)$ are projective, that $A'A$, $A'B$, $A'B'$ are self-corresponding rays of these two pencils and hence all the corresponding rays coincide. Therefore, $A'$, $E$, $H$ lie on a straight line, etc.




**2757 [1919, 124]. Proposed by E. P. Lane, Rice Institute, Houston, Texas.**

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + y^3 = 0.$$

## Solution by Alexander Dillingham, U. S. Naval Academy.

Interchanging the variables and setting $q = dx/dy$, we have $d^2y/dx^2 = -(dq/dy)/q^3$. The given equation becomes, after making these substitutions,

$$\frac{dq}{dy} + 3yq^2 - y^3q^3 = 0.$$

By inspection we find a particular integral $q_1 = y^{-2}$ and hence we are led to put $q = q_1 + v$, where $v$ is a function of $y$. The equation (1) now becomes a Bernoulli equation $dv/dy + 3v/y = v^3y^3$ which reduces, on putting $z = v^{-2}$, to the linear equation $dz/dy - 6z/y = -2y^3$ with the integrating factor $y^{-6}$. We then obtain $zy^{-6} = \int(-2y^{-3})dy = y^{-2} + c_1$ or $z = y^4 + c_1y^6 = v^{-2}$. Hence we have in turn

$$v = \pm\frac{1}{y^2\sqrt{1+c_1y^2}}, \qquad q = \frac{1}{y^2} \pm \frac{1}{y^2\sqrt{1+c_1y^2}} = \frac{dx}{dy},$$

and by integrating the last equation, we have finally

$$x = -\frac{1}{y} \mp \sqrt{y^{-2}+c_1} + c_2.$$

Also solved by R. D. Bohannan, P. J. da Cunha, E. B. Escott, M. Gutten, H. Halperin, H. L. Olson, and Elijah Swift.

## NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

At the Carnegie Institute of Technology Assistant Professor J. R. EVERETT, of Baker University, and Professor G. W. HESS, of Bethany College, have been appointed instructors in mathematics.

Dr. F. ELIZABETH LE STOURGEON, of Carleton College, has accepted a position as assistant professor of mathematics at the University of Kentucky.

Professor EUGENE TAYLOR, of the University of Wisconsin, has been appointed head of the department of mathematics at the University of Idaho.

Dr. R. B. ROBBINS, of the University of Michigan, has been promoted to an assistant professorship. He has been granted leave of absence for the year, which he will spend in practical study in statistics and insurance offices.

Professor J. MCMAHON, of Cornell University, has a sabbatical leave of absence for the year, and will retire at its close.

Professor P. H. GRAHAM, of Agnes Scott College, has been appointed instructor in mathematics in Washington Square College, New York University.

Instructor R. M. MATHEWS, of the University of Minnesota, has been appointed assistant professor of mathematics at Wesleyan University.

Mrs. MAYME I. LOGSDON, instructor in mathematics at Northwestern University, but more recently fellow at the University of Chicago, has been appointed associate in mathematics at the University of Chicago.

At Wellesley College Doctors MARY F. CURTIS and LENNIE P. COPELAND have been promoted to assistant professorships in mathematics.

At Harvard University the following new instructors in mathematics have been appointed for the year 1920–21: Assistant Professor C. A. GARABEDIAN of New Hampshire College; Messrs. R. E. LANGER, E. L. MACKIE and HARRY LEVY of Harvard University; and Mr. A. J. COOK of University of Alberta.

At the University of Minnesota, Professor W. H. BUSSEY has been promoted to the rank of professor and made assistant dean in the College of Science, Literature and the Arts; assistant professors A. L. UNDERHILL, R. W. BRINK, and W. L. HART have been promoted to associate professorships; F. K. BAIER, JR., of Pennsylvania State College, and Dr. GLADYS GIBBENS have been appointed instructors in mathematics.

At the University of Wisconsin, Professor C. S. SLICHTER has been appointed dean of the Graduate School to succeed Professor G. C. COMSTOCK; Associate Professor E. B. SKINNER has been promoted to a full professorship; Assistant Professor H. W. MARCH has been promoted to an associate professorship; Dr. WARREN WEAVER, of the California Institute of Technology, has been appointed to an assistant professorship; and Messrs. HAROLD DAVIS, of Harvard, and M. L. MACQUEEN, of Southwestern Presbyterian University, have been appointed instructors in mathematics.

At the University of Paris, Dr. P. PAINLEVÉ, professor of rational mechanics and a former premier of France, has been appointed professor of celestial mechanics in place of Professor P. APPELL. Dr. E. J. CARTAN, professor of differential calculus has been appointed to Professor Painlevé's chair, and Professor E. P. J. VESSIOT, new sub-director of the Ecole Normale Supérieure [*1920*, 338], to Professor Cartan's. Professors PAINLEVÉ and E. BOREL have gone to China on an educational mission; the former was in Washington last May.

Mr. A. E. JOLLIFFE, tutor in mathematics at Corpus Christi College, Oxford, has been appointed to the University chair of mathematics tenable at the Royal Holloway College, London.

Miss MARY A. COLPITTS, instructor in mathematics at the University of Wisconsin [*1919*, 419] died at Madison on July 11, aged twenty-eight years.

Dr. J. N. STOCKWELL, of Cleveland, O., author of contributions to mathematical astronomy, died on May 18, 1920, aged eighty-eight years.

*Popular Astronomy* reports that Dr. A. BERBERICH, the noted German astronomer, who had done much valuable work in the computation of comet and asteroid orbits, died on April 27, 1920.

*Nature* reports the death on June 20, 1920, of Dr. F. A. TARLETON of the board of Trinity College, Dublin, aged seventy-nine years. He was professor of natural philosophy at Trinity College from 1890 to 1901. His books, *An Introduction to the Mathematical Theory of Attractions* (1899, vol. 2, 1913), and *An Elementary Treatise on Dynamics*, in collaboration with B. Williamson (1885, third edition, 1900), were widely used in Great Britain. "As professor of natural philosophy, Dr. Tarleton followed the traditions of his distinguished predecessors, Williamson, Townsend, and Jellett, in treating the subject from a strictly mathematical point of view. Although he had considerable practical acquaintance with experimental science, he flatly ignored the judicial aphorisms of Francis Bacon and, instead of treating mathematics as the handmaid of physics, he rather inverted the order, and almost succeeded in reducing hydrodynamics, elasticity, magnetism, and electricity to branches of pure mathematics." (R. A. P. Rogers).

At the last Yale commencement in conferring the degree of Master of Arts on Dean HAWKES, of Columbia University, President Hadley said: "Herbert Edwin Hawkes: B.A., Yale, 1896; Ph.D., 1900; like many of his classmates, Dr. Hawkes became a member of the Yale faculty, and taught mathematics for twelve years. In 1910 he was called to Columbia as professor; he was such a conspicuous success in administration that he was made dean of the college. He is the author of books in his chosen field, but his chief distinction is a worker of miracles—he has made hundreds of young men love mathematics. Perhaps they would not love mathematics so much if they did not love him even more. A living force in education."

At a meeting of the Academy of Sciences of the Institute of France on May 17, 1920, (1) a committee of seven (the president of the academy and three representatives each from the divisions of physical and mathematical sciences) was appointed to present a list of candidates for the chair of General History of Sciences in the Collège de France. (2) Professor L. E. DICKSON, of the University of Chicago, was elected (41 votes out of 45) "correspondant" of the Academy, in the section of geometry. Two votes were cast for G. CASTELNUOVO and two for E. I. FREDHOLM.

By a decree of 1909 the number of correspondants of the Academy of Sciences was increased to 116, 10 for each section except astronomy where there are 16, The other sections are: geometry, mechanics, physics, geography and navigation, chemistry, mineralogy, botany, rural economics, anatomy and zoology, medicine and surgery. Of the 50 correspondants in the section of geometry since 1816 only six have been English or American, namely: Ivory (elected in 1828), Hamilton (1844), Sylvester (1863), Spottiswoode (1876), Salmon (1884), and Dickson (1920).

By way of celebrating Professor Dickson's election colleagues and friends in the University of Chicago tendered him a complimentary luncheon. Professor A. A. Michelson, the only other "correspondant" of the Academy of Sciences at the University, presided.

Professor Dickson's productive activity during the twenty years of his connection with the mathematics department of the University of Chicago has been notable. In *Publications of the Members of the University 1902–1916* (Chicago, 1917), 6 of his books (one in collaboration with Miller and Blichfeldt) and 123 of his papers are listed. Of his 29 papers published in this MONTHLY 14 only are here mentioned; the others appeared before 1902. Since 1916 have appeared the first two volumes of his monumental *History of the Theory of Numbers* (cf. *1919*, 396–403). The second volume (25–803 pages) was published in August.

Professor Dickson was managing editor of this MONTHLY 1902–1906 and associate editor, 1906–1908. He was associate editor of the *Transactions of the American Mathematical Society*, 1902–1910 and joint editor 1911–1916. He has been president of the American Mathematical Society, is a member of the Division of Physical Sciences of the National Research Council, and is chairman of the American Section of the International Mathematical Union whose membership was listed in our last issue (*1920*, 340).

## THE FIFTH SUMMER MEETING OF THE ASSOCIATION.

The fifth summer meeting of the Mathematical Association of America was held at the University of Chicago on Monday, September 6, 1920, in conjunction with, and immediately preceding, the summer meeting and colloquium of the American Mathematical Society. 132 were present at the meeting, including the following 114 members of the Association:

O. W. ALBERT, Grinnell College.
EDNA ALLEN, Chicago, Ill.
R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.

P. M. BATCHELDER, University of Texas.
I. A. BARNETT, University of Saskatchewan.
SUZAN R. BENEDICT, Smith College.
A. A. BENNETT, University of Texas.
G. D. BIRKHOFF, Harvard University.
VEVIA BLAIR, Horace Mann High School.
G. A. BLISS, University of Chicago.
R. L. BORGER, Ohio University.
J. W. BRADSHAW, University of Michigan.
W. H. BUSSEY, University of Minnesota.

W. D. CAIRNS, Oberlin College.
D. F. CAMPBELL, Armour Institute of Technology.
J. W. CAMPBELL, University of Alberta.
A. L. CANDY, University of Nebraska.
J. A. CAPARO, Notre Dame University.
F. E. CARR, Oberlin College.
W. E. CEDERBERG, University of Wisconsin.
E. W. CHITTENDEN, University of Iowa.
L. M. COFFIN, Coe College.
A. R. CRATHORNE, University of Illinois.
D. R. CURTISS, Northwestern University.

H. H. DALAKER, University of Minnesota.
Sister MARIOLA DOBBIN, St. Clara College.
E. L. DODD, University of Texas.
J. E. DOTTERER, Manchester College.
L. W. DOWLING, University of Wisconsin.
OTTO DUNKEL, Washington University.

M. D. EARLE, Furman University.
G. C. EVANS, Rice Institute.
H. S. EVERETT, Bucknell University.

ZOE FERGUSON, Crane Junior College.
B. F. FINKEL, Drury College.
L. R. FORD, Rice Institute.
W. B. FORD, University of Michigan.

M. G. GABA, University of Nebraska.
C. D. GARLOUGH, Wheaton College.
W. H. GARRETT, Baker University.
D. C. GILLESPIE, Cornell University.
R. E. GILMAN, Brown University.
CORNELIUS GOUWENS, State College of Iowa.
C. F. GUMMER, Queen's University.

W. L. HART, University of Minnesota.
M. W. HASKELL, University of California.
OLIVE C. HAZLETT, Mount Holyoke College.
E. R. HEDRICK, University of Missouri.
ALBERT HEINZ, Tsing Hua College.
T. H. HILDEBRANDT, University of Michigan.
L. A. HOPKINS, University of Michigan.
JEWELL C. HUGHES, University of Arkansas.
W. A. HURWITZ, Cornell University.

DUNHAM JACKSON, University of Minnesota.
G. H. JAMISON, Kirksville (Mo.) State Normal School.

CLARIBEL KENDALL, University of Colorado.
S. D. KILLAM, University of Alberta.
J. M. KINNEY, Hyde Park (Chicago) High School.
H. W. KUHN, Ohio State University.

GILLIE A. LAREW, Randolph-Macon Woman's College.
D. A. LEHMAN, Goshen College.
FLORA E. LE STOURGEON, University of Kentucky.
MAYME I. LOGSDON, University of Chicago.
A. C. LUNN, University of Chicago.
E. B. LYTLE, University of Illinois.

S. H. MACDONALD, Colorado Agricultural College.
L. E. MCCARTY, Michigan College of Mines.
R. B. MCCLENON, Grinnell College.
J. V. MCKELVEY, Iowa State College.
HELEN A. MERRILL, Wellesley College.
BESSIE I. MILLER, Rockford College.